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The $O(\alpha_s^2)$ heavy quark corrections to charged current deep-inelastic scattering at large virtualities

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Abstract

We calculate the $O(\alpha_s^2)$ heavy flavor corrections to charged current deep-inelastic scattering at large scales $Q^2 \gg m^2$. The contributing Wilson coefficients are given as convolutions between massive operator matrix elements and massless Wilson coefficients. Foregoing results in the literature are extended and corrected. Numerical results are presented for the kinematic region of the HERA data.

1. Introduction

The heavy flavor corrections to deep-inelastic scattering obey different scaling violations both in neutral and charged current scattering if compared to the massless contributions [1]. Furthermore, for charged current reactions these contributions constitute out of flavor excitation on the one hand, e.g. $s \to c$ transitions, and also heavy quark pair production at higher orders in the coupling constant $\alpha_s(M_Z^2)$. In the charged current case most of the data are situated at higher values of Q^2, cf. [2,3]. Therefore the representation of the heavy flavor Wilson coefficients in the region $Q^2 \gg m^2$ can be obtained using the factorization [4] into massive operator matrix

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elements (OMEs) and the massless Wilson coefficients [5–10]. In the past a series of analytic results has been calculated for neutral current reactions in this way [11–22].

In the present paper we calculate the $O(\alpha_s^2)$ corrections in the charged current case, extending and correcting Ref. [23]. The $O(\alpha_s)$ corrections were computed in [24–26] before. In [23] the heavy flavor Wilson coefficients $H_{2,g}^{W,(2)}$, $H_{2,q}^{\mathrm{PS},(2)}$, $H_{3,g}^{W,\mathrm{PS},(2)}$ and $H_{3,q}^{\mathrm{PS},(2)}$ were calculated. Since this was not the complete set we also calculate the remaining Wilson coefficients and compare the present results with the previous ones. The heavy flavor Wilson coefficients to $O(\alpha_s^2)$ will allow to refine QCD fits w.r.t. the extraction of the individual sea quarks, in particular also the strange quark distribution, cf. [27,28].

The paper is organized as follows. We give first a summary of the charged current structure functions with emphasis on the heavy flavor contributions and present the general structure of the different heavy flavor Wilson coefficients in the limit $Q^2 \gg m^2$. Here combinations which are invariant under current crossing are important to allow for proper renormalization. In Section 3 the Wilson coefficients are presented in Mellin-N space to $O(\alpha_s^2)$. Numerical results are given in Section 4 and Section 5 contains the conclusions. In Appendices A–C technical aspects are dealt with and we also present the Wilson coefficients in x-space there.

2. The structure functions

The scattering cross sections for charged current deep-inelastic lepton–nucleon scattering are parameterized by the three structure functions F_1, F_2, F_3:

$$\frac{d\sigma^{\nu(\bar{\nu})}}{dx\,dy} = \frac{G_F^2 s}{4\pi}\frac{M_W^4}{(M_W^2+Q^2)^2}\left\{\left(1+(1-y)^2\right)F_2^{W^\pm} - y^2 F_L^{W^\pm} \pm \left(1-(1-y)^2\right)xF_3^{W^\pm}\right\}, \tag{2.1}$$

$$\frac{d\sigma^{e^-(e^+)}}{dx\,dy} = \frac{G_F^2 s}{4\pi}\frac{M_W^4}{(M_W^2+Q^2)^2}\left\{\left(1+(1-y)^2\right)F_2^{W^\mp} - y^2 F_L^{W^\mp} \pm \left(1-(1-y)^2\right)xF_3^{W^\mp}\right\}, \tag{2.2}$$

where

$$F_L = F_2 - 2xF_1. \tag{2.3}$$

Here $x = Q^2/(sy)$ and y denote the Bjorken variables, Q^2 is the virtuality of the exchanged electro-weak gauge boson, s is the cms-energy squared, M_W is the mass of the $W^\pm$-bosons, and G_F is Fermi's constant.

At Born level the structure functions are given by the following combinations of parton distribution functions (PDFs) $q \equiv q(x, Q^2)$, cf. [29]:

$$F_2^{W^+} = 2x\left[\left(|V_{ud}|^2+|V_{cd}|^2\right)d + \left(|V_{us}|^2+|V_{cs}|^2\right)s + \left(|V_{ud}|^2+|V_{us}|^2\right)\bar{u}\right], \tag{2.4}$$

$$F_2^{W^-} = 2x\left[\left(|V_{ud}|^2+|V_{cd}|^2\right)\bar{d} + \left(|V_{us}|^2+|V_{cs}|^2\right)\bar{s} + \left(|V_{ud}|^2+|V_{us}|^2\right)u\right], \tag{2.5}$$

$$xF_3^{W^+} = 2x\left[\left(|V_{ud}|^2+|V_{cd}|^2\right)d + \left(|V_{us}|^2+|V_{cs}|^2\right)s - \left(|V_{ud}|^2+|V_{us}|^2\right)\bar{u}\right], \tag{2.6}$$

$$xF_3^{W^-} = 2x\left[-\left(|V_{ud}|^2+|V_{cd}|^2\right)\bar{d} - \left(|V_{us}|^2+|V_{cs}|^2\right)\bar{s} + \left(|V_{ud}|^2+|V_{us}|^2\right)u\right], \tag{2.7}$$

$$F_L^{W^+} = F_L^{W^-} = 0, \tag{2.8}$$

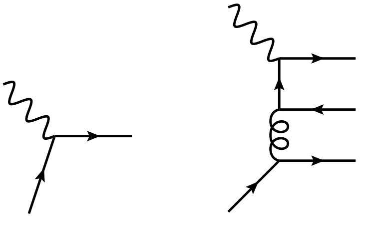

Fig. 1. Born diagrams for main processes contributing to W-boson exchange. The wavy lines denote W bosons, the curly lines denote gluons, and the arrow-lines denote quarks.

where V_{ij} denote the Cabibbo–Kobayashi–Maskawa matrix elements [30,31]. In the following we refer to the four-quark picture.

It is worthwhile to study combinations of cross sections

$$\frac{d\sigma^{\nu}}{dx\,dy} + \frac{d\sigma^{\bar\nu}}{dx\,dy} =: \frac{G_F^2 s}{4\pi}\Big\{\big(1+(1-y)^2\big)F_2^{W^+ + W^-} - y^2 F_L^{W^+ + W^-} + \big(1-(1-y)^2\big)xF_3^{W^+ + W^-}\Big\}, \tag{2.9}$$

$$\frac{d\sigma^{\nu}}{dx\,dy} - \frac{d\sigma^{\bar\nu}}{dx\,dy} =: \frac{G_F^2 s}{4\pi}\Big\{\big(1+(1-y)^2\big)F_2^{W^+ - W^-} - y^2 F_L^{W^+ - W^-} + \big(1-(1-y)^2\big)xF_3^{W^+ - W^-}\Big\}, \tag{2.10}$$

which are symmetric/antisymmetric under crossing, respectively. The following partonic quantities are introduced:

$$\mathcal{F}_2^{W^\pm} := \frac{1}{2x}F_2^{W\pm}, \qquad \mathcal{F}_3^{W^\pm} := \frac{1}{2}F_3^{W\pm}. \tag{2.11}$$

The Mellin transforms of the structure functions read

$$F_2^{W^\pm}(N) := \int_0^1 dx\, x^{N-2} F_2^{W^\pm}(x) = 2\int_0^1 dx\, x^{N-1}\mathcal{F}_2^{W^\pm}(x) =: 2\mathcal{F}_2^{W^\pm}(N),$$
$$F_3^{W^\pm}(N) := \int_0^1 dx\, x^{N-1} F_3^{W^\pm}(x) = 2\int_0^1 dx\, x^{N-1}\mathcal{F}_3^{W^\pm}(x) =: 2\mathcal{F}_3^{W^\pm}(N). \tag{2.12}$$

In the following formulae, we will work in the Mellin space and drop the argument N for brevity.

There are diagrams in which the incoming fermion line runs through the W-boson–quark vertex, and others where these two fermion lines are separated. Examples are given in Fig. 1.

It is useful to separate the corresponding terms in the Wilson coefficients into "valence" and "sea" contributions, respectively. The valence parts are flavor-diagonal while the sea parts do not distinguish different flavors. However, differences in the quark masses are detected. Hence the c-quark is treated differently from u, d, s.

Obviously, all terms built from graphs like Fig. 1(a) and their QCD corrections form valence contributions, and all sea contributions are built from graphs like Fig. 1(b). However, there are interference contributions from the latter class of graphs, see e.g. Fig. 2, which clearly form valence terms.

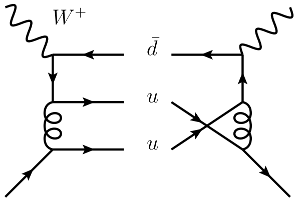

Fig. 2. Valence like interference terms for W^+u-scattering and $W^+\bar{d}$-scattering, which have no counter parts on tree level.

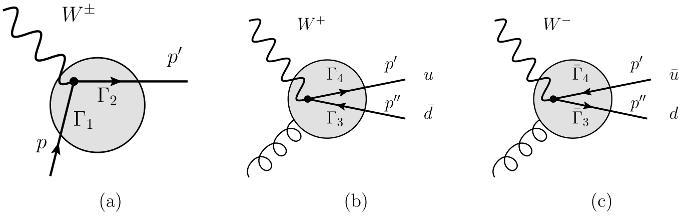

Fig. 3. The QCD corrections, denoted by the gray area, are connected to the scattered quark line through gluon exchange.

Note that minus signs derive from the charge conjugation antisymmetry of the fermion line to which the W-boson is attached. This antisymmetry is due to the presence of a single γ_5-matrix and hence only occurs in contributions to F_3. The emergence of these minus signs is shortly illustrated in the following. In these considerations, factors of i or (-1) stemming from the Feynman rules are not of relevance, since expressions with the same number of vertices and propagators are compared.

Fig. 3 schematically shows the structure of the graphs in which (a) the fermion line coupling to the $W^\pm qq$-vertex is incoming or in which (b, c) this fermion is pair produced. The incoming gluon line in the latter could also be replaced by an incoming fermion line, that passes through to the final state. The gray area denotes any QCD correction, that couples to the fermion line via the gluon field. The fermion trace of a diagram depicted by Fig. 3(a) can be written as

$$T_q^{W^+,V} = \sum_{\text{Spins}} \bar{u}(p')\Gamma_2 \frac{1+\gamma_5}{2}\gamma_\mu \Gamma_1 u(p)\bar{u}(p)\bar{\Gamma}_1 \frac{1+\gamma_5}{2}\gamma_\nu \bar{\Gamma}_2 u(p'), \tag{2.13}$$

where the Γ_i denote products of Dirac matrices, multiplied by real numbers which also include the denominators of the propagators, and $\bar{\Gamma}_i = \gamma_0 \Gamma_i^\dagger \gamma_0$ is just Γ_i with inverted order of the factors. Due to the charge conjugation properties of the Dirac matrices and Dirac bispinors, there is a bijection onto diagrams with an antifermion in the initial state. Assuming an antifermion in the initial state, the same trace has the form

$$\begin{aligned} T_{\bar{q}}^{W^+,V} &= \sum_{\text{Spins}} \bar{v}(p)\bar{\Gamma}_1 \frac{1+\gamma_5}{2}\gamma_\mu \bar{\Gamma}_2 v(p')\bar{v}(p')\Gamma_2 \frac{1+\gamma_5}{2}\gamma_\nu \Gamma_1 v(p) \\ &= \sum_{\text{Spins}} \bar{v}(p')\Gamma_2 \frac{1+\gamma_5}{2}\gamma_\nu \Gamma_1 v(p)\bar{v}(p)\bar{\Gamma}_1 \frac{1+\gamma_5}{2}\gamma_\mu \bar{\Gamma}_2 v(p'). \end{aligned} \tag{2.14}$$

Since the difference between fermion and antifermion bispinors in the trace only affects the part $\propto m^2$, it contributes to the power corrections only, and due to antisymmetry of the γ_5 part, $T_{\bar{q}}^{W^+}$

and $T_q^{W^+}$ only differ by a minus sign in front of γ_5. This leads to the minus signs in Eqs. (2.21), (2.22) below when compared to (2.19), (2.20).

In case of the "sea"-contributions depicted in Fig. 3(b) and (c) the traces read:

$$T^{W^+,\mathrm{S}} = \sum_{\mathrm{Spins}} \bar{u}_u(p')\Gamma_4 \frac{1+\gamma_5}{2}\gamma_\mu \Gamma_3 v_d(p'')\bar{v}_d(p'')\bar{\Gamma}_3 \frac{1+\gamma_5}{2}\gamma_\nu \bar{\Gamma}_4 u_u(p'), \tag{2.15}$$

$$\begin{aligned} T^{W^-,\mathrm{S}} &= \sum_{\mathrm{Spins}} \bar{u}_d(p'')\bar{\Gamma}_3 \frac{1+\gamma_5}{2}\gamma_\mu \bar{\Gamma}_4 v_u(p')\bar{v}_u(p')\Gamma_4 \frac{1+\gamma_5}{2}\gamma_\nu \Gamma_3 u_d(p'') \\ &= \sum_{\mathrm{Spins}} \bar{v}_u(p')\Gamma_4 \frac{1+\gamma_5}{2}\gamma_\nu \Gamma_3 u_d(p'')\bar{u}_d(p'')\bar{\Gamma}_3 \frac{1+\gamma_5}{2}\gamma_\mu \bar{\Gamma}_4 v_u(p'). \end{aligned} \tag{2.16}$$

Here, bispinors of down-type (anti)quarks are marked by the subscript d and the ones of up-type (anti)quarks are marked by u. In the complete contribution one will find a corresponding diagram with Γ_4 and $\bar{\Gamma}_3$ interchanged, if the down- and up-type lines have the same mass or are both massless. This leads to the symmetry between $T^{W^+,\mathrm{S}}$ and $T^{W^-,\mathrm{S}}$ when summed over all diagrams. The combination of this symmetry and the antisymmetry from above leads to the relations [23]:

$$C_{3,q}^{W,\mathrm{PS}} = C_{3,g}^{W} = 0 \quad \text{and} \quad L_{3,q}^{W,\mathrm{PS}} = L_{3,g}^{W} = 0. \tag{2.17}$$

Another source for negative signs are the valence like interference terms. While the coupling of W^+ with the down-type quarks in the non-singlet channel is described by (QCD corrections to) diagrams like Fig. 1(a), the non-singlet coupling of W^- to a down-type quark is only possible through the interference terms of Fig. 2. Calling these different contributions $C_d^{W^+,\mathrm{NS}}$ and $C_d^{W^-,\mathrm{NS}}$, the Wilson coefficients of the combinations of W^+ and W^- take the form:

$$C_d^{W^-+W^-,\mathrm{NS}} = C_d^{W^+,\mathrm{NS}} + C_d^{W^-,\mathrm{NS}}, \qquad C_d^{W^--W^-,\mathrm{NS}} = C_d^{W^+,\mathrm{NS}} - C_d^{W^-,\mathrm{NS}}. \tag{2.18}$$

As a consequence these combinations are even/odd under crossing, but the individual $W^\pm$-contributions are not (cf. [32]). Interestingly, in the Mellin space this lack of symmetry is reflected in an oscillating behavior characterized by a factor $(-1)^N$ which, by Carlson's theorem [33–35], prevents the Wilson coefficients from being Mellin invertible.

For these reasons it is useful to study combinations of structure functions, which have a crossing symmetry by construction. The factorization of these combinations reads:

$$\begin{aligned} \mathcal{F}_2^{W^+} + \mathcal{F}_2^{W^-} = {} & \big(|V_{du}|^2(d+\bar{d}) + |V_{su}|^2(s+\bar{s}) + V_u(u+\bar{u})\big) \\ & \times \big(C_{2,q}^{W^++W^-,\mathrm{NS}} + L_{2,q}^{W^++W^-,\mathrm{NS}}\big) \\ & + \big(|V_{dc}|^2(d+\bar{d}) + |V_{sc}|^2(s+\bar{s})\big) H_{2,q}^{W^++W^-,\mathrm{NS}} \\ & + 2V_u\big[\big(C_{2,q}^{W,\mathrm{PS}} + L_{2,q}^{W,\mathrm{PS}}\big)\Sigma + \big(C_{2,g}^{W} + L_{2,g}^{W}\big)G\big] \\ & + 2V_c\big[H_{2,q}^{W,\mathrm{PS}}\Sigma + H_{2,g}^{W}G\big], \end{aligned} \tag{2.19}$$

$$\begin{aligned} \mathcal{F}_2^{W^+} - \mathcal{F}_2^{W^-} = {} & \big(|V_{du}|^2(d-\bar{d}) + |V_{su}|^2(s-\bar{s}) - V_u(u-\bar{u})\big) \\ & \times \big(C_{2,q}^{W^+-W^-,\mathrm{NS}} + L_{2,q}^{W^+-W^-,\mathrm{NS}}\big) \\ & + \big(|V_{dc}|^2(d-\bar{d}) + |V_{sc}|^2(s-\bar{s})\big) H_{2,q}^{W^+-W^-,\mathrm{NS}}, \end{aligned} \tag{2.20}$$

$$\begin{aligned}
\mathcal{F}_3^{W^+} + \mathcal{F}_3^{W^-} = & \big(|V_{du}|^2(d-\bar{d}) + |V_{su}|^2(s-\bar{s}) + V_u(u-\bar{u})\big) \\
& \times \big(C_{3,q}^{W^+-W^-,\mathrm{NS}} + L_{3,q}^{W^+-W^-,\mathrm{NS}}\big) \\
& + \big(|V_{dc}|^2(d-\bar{d}) + |V_{sc}|^2(s-\bar{s})\big) H_{3,q}^{W^+-W^-,\mathrm{NS}},
\end{aligned} \tag{2.21}$$

$$\begin{aligned}
\mathcal{F}_3^{W^+} - \mathcal{F}_3^{W^-} = & \big(|V_{du}|^2(d+\bar{d}) + |V_{su}|^2(s+\bar{s}) - V_u(u+\bar{u})\big) \\
& \times \big(C_{3,q}^{W^++W^-,\mathrm{NS}} + L_{3,q}^{W^++W^-,\mathrm{NS}}\big) \\
& + \big(|V_{dc}|^2(d+\bar{d}) + |V_{sc}|^2(s+\bar{s})\big) H_{3,q}^{W^++W^-,\mathrm{NS}} \\
& + 2V_c\big[H_{3,q}^{W,\mathrm{PS}} \Sigma + H_{3,g}^{W} G\big],
\end{aligned} \tag{2.22}$$

with

$$V_u = |V_{du}|^2 + |V_{su}|^2, \tag{2.23}$$

$$V_d = |V_{du}|^2 + |V_{dc}|^2, \tag{2.24}$$

$$V_s = |V_{su}|^2 + |V_{sc}|^2, \tag{2.25}$$

$$V_c = |V_{dc}|^2 + |V_{sc}|^2. \tag{2.26}$$

As a result, either even or odd moments contribute to the combinations in the Mellin space. One finds [32,36–38]:

$$F_2^{W^+} + F_2^{W^-}: \text{ even } N, \tag{2.27}$$

$$F_2^{W^+} - F_2^{W^-}: \text{ odd } N, \tag{2.28}$$

$$F_3^{W^+} + F_3^{W^-}: \text{ odd } N, \tag{2.29}$$

$$F_3^{W^+} - F_3^{W^-}: \text{ even } N. \tag{2.30}$$

These sequences of even or odd moments then have well defined x-space counter parts.

In order to derive factorization formulae, we choose to take a safe detour via the relations of parton distributions in the variable flavor number scheme $(q', \bar{q}')$ [12,18]:

$$\begin{aligned}
& q' + \bar{q}' = A_{qq,Q}^{\mathrm{NS}}(q+\bar{q}) + \tilde{A}_{qq,Q}^{\mathrm{PS}} \Sigma + \tilde{A}_{qg,Q} G, \\
& c' + \bar{c}' = A_{Qq}^{\mathrm{PS}} \Sigma + A_{Qg} G, \\
& \Sigma' = \big(n_f \tilde{A}_{qq,Q}^{\mathrm{PS}} + A_{Qq}^{\mathrm{PS}} + A_{qq,Q}^{\mathrm{NS}}\big)\Sigma + (n_f \tilde{A}_{qg,Q} + A_{Qg})G, \\
& G' = A_{gq,Q}\Sigma + A_{gg,Q} G, \\
& \Delta_q' = q' + \bar{q}' - \frac{1}{n_f+1}\Sigma'.
\end{aligned} \tag{2.31}$$

Here the following notation was used:

$$\tilde{A}_{ij}(n_f+1) \equiv \frac{1}{n_f} A_{ij}(n_f+1). \tag{2.32}$$

From this point on, the number of light flavors contributing to the light flavor Wilson coefficients is written explicitly as an argument. The four-flavor expressions read:

$$\mathcal{F}_2^{W^+}+\mathcal{F}_2^{W^-}=\big(V_d\Delta_d'+V_s\Delta_s'+V_u\Delta_u'+V_c\Delta_c'\big)C_{2,q}^{W^++W^-,\mathrm{NS}}(n_f+1)$$
$$+\frac{V_u+V_c}{n_f+1}\big[C_{2,q,(n_f+1)}^{W^++W^-,\mathrm{S}}\Sigma'+2(n_f+1)C_{2,g}^{W}(n_f+1)G'\big], \quad (2.33)$$
$$\mathcal{F}_2^{W^+}-\mathcal{F}_2^{W^-}=\big(V_d\big(d'-\bar{d}'\big)+V_s\big(s'-\bar{s}'\big)-V_u\big(u'-\bar{u}'\big)-V_c\big(c'-\bar{c}'\big)\big)$$
$$\times C_{2,q}^{W^+-W^-,\mathrm{NS}}(n_f+1), \quad (2.34)$$
$$\mathcal{F}_3^{W^+}-\mathcal{F}_3^{W^-}=\big(V_d\Delta_d'+V_s\Delta_s'-V_u\Delta_u'-V_c\Delta_c'\big)C_{3,q}^{W^++W^-,\mathrm{NS}}(n_f+1), \quad (2.35)$$
$$\mathcal{F}_3^{W^+}+\mathcal{F}_3^{W^-}=\big(V_d\big(d'-\bar{d}'\big)+V_s\big(s'-\bar{s}'\big)+V_u\big(u'-\bar{u}'\big)+V_c\big(c'-\bar{c}'\big)\big)$$
$$\times C_{3,q}^{W^+-W^-,\mathrm{NS}}(n_f+1). \quad (2.36)$$

Comparing the coefficients of Σ, G, Δ_q, V_u, V_c in these relations with the 3-flavor representation (2.19) one finds

$$C_{2,q}^{W^+\pm W^-,\mathrm{NS}}(n_f)+L_{2,q}^{W^+\pm W^-,\mathrm{NS}}=A_{qq,Q}^{\mathrm{NS}}C_{2,q}^{W^+\pm W^-,\mathrm{NS}}(n_f+1),$$
$$H_{2,q}^{W^+\pm W^-,\mathrm{NS}}=A_{qq,Q}^{\mathrm{NS}}C_{2,q}^{W^+\pm W^-,\mathrm{NS}}(n_f+1),$$
$$C_{2,q}^{W,\mathrm{PS}}(n_f)+L_{2,q}^{W,\mathrm{PS}}=\tilde{A}_{qq,Q}^{\mathrm{PS}}C_{2,q}^{W^++W^-,\mathrm{NS}}(n_f+1)$$
$$+C_{2,q}^{W,\mathrm{PS}}(n_f+1)\big(n_f\tilde{A}_{qq,Q}^{\mathrm{PS}}+A_{Qq}^{\mathrm{PS}}+A_{qq,Q}^{\mathrm{NS}}\big)$$
$$+A_{gq,Q}C_{2,g}^{W}(n_f+1),$$
$$H_{2,q}^{W,\mathrm{PS}}=\frac{1}{2}\big(\tilde{A}_{qq,Q}^{\mathrm{PS}}+A_{Qq}^{\mathrm{PS}}\big)C_{2,q}^{W^++W^-,\mathrm{NS}}(n_f+1)$$
$$+\big(n_f\tilde{A}_{qq,Q}^{\mathrm{PS}}+A_{Qq}^{\mathrm{PS}}+A_{qq,Q}^{\mathrm{NS}}\big)C_{2,q}^{W,\mathrm{PS}}(n_f+1)+A_{gq,Q}C_{2,g}^{W}(n_f+1),$$
$$C_{2,g}^{W}(n_f)+L_{2,g}^{W}=\tilde{A}_{qg,Q}C_{2,q}^{W^++W^-,\mathrm{NS}}(n_f+1)$$
$$+(n_f\tilde{A}_{qg,Q}+A_{Qg})C_{2,q}^{W,\mathrm{PS}}(n_f+1)+A_{gg,Q}C_{2,g}^{W}(n_f+1),$$
$$H_{2,g}^{W}=\frac{1}{2}(\tilde{A}_{qg,Q}+A_{Qg})C_{2,q}^{W^++W^-,\mathrm{NS}}(n_f+1)$$
$$+(n_f\tilde{A}_{qg,Q}+A_{Qg})C_{2,q}^{W,\mathrm{PS}}(n_f+1)+A_{gg,Q}C_{2,g}^{W}(n_f+1), \quad (2.37)$$

where the odd-N combinations are included in analogy to the even-N ones. From Eqs. (2.35) and (2.22) one can deduce similarly

$$L_{3,q}^{W^+\pm W^-,\mathrm{NS}}=A_{qq,Q}^{\mathrm{NS}}C_{3,q}^{W^+\pm W^-,\mathrm{NS}}(n_f+1)-C_{3,q}^{W^+\pm W^-,\mathrm{NS}}(n_f),$$
$$H_{3,q}^{W^+\pm W^-,\mathrm{NS}}=A_{qq,Q}^{\mathrm{NS}}C_{3,q}^{W^+\pm W^-,\mathrm{NS}}(n_f+1),$$
$$H_{3,q}^{W,\mathrm{PS}}=\frac{1}{2}\big(\tilde{A}_{qq,Q}^{\mathrm{PS}}-A_{Qq}^{\mathrm{PS}}\big)C_{3,q}^{W^++W^-,\mathrm{NS}}(n_f+1),$$
$$H_{3,g}^{W}=\frac{1}{2}(\tilde{A}_{qg,Q}-A_{Qg})C_{3,q}^{W^++W^-,\mathrm{NS}}(n_f+1). \quad (2.38)$$

By inserting the odd-N factorization relations into (2.20) and (2.21), and comparing with (2.34) and (2.36), respectively, one finds:

$$
\begin{aligned}
&q' - \bar{q}' = A_{qq,Q}^{\mathrm{NS}}(q - \bar{q}), \\
&c' - \bar{c}' = 0.
\end{aligned}
\tag{2.39}
$$

Expanding the above relations up to order a_s^2,

$$
f(a_s) = \sum_{l=0}^{\infty} a_s^l f^{(l)}, \tag{2.40}
$$

one finds the asymptotic representations. The relations for the longitudinal structure function F_L are almost complete analogs to the ones for F_2, so they are included using the index $i = 2/L$, where the only structural difference, denoted by Kronecker symbols $\delta_{i,2}$, derives from the fact, that the coefficients C_L^{NS} do not have a Born contribution. On the Born level, one obviously has

$$
\begin{aligned}
&H_{i,q}^{W^+ \pm W^-,\mathrm{NS},(0)} = \delta_{i,2}, \\
&H_{3,q}^{W^+ \pm W^-,\mathrm{NS},(0)} = 1.
\end{aligned}
\tag{2.41}
$$

At 1-loop level, one obtains

$$
\begin{aligned}
&H_{i,q}^{W^+ \pm W^-,\mathrm{NS},(1)} = C_{i,q}^{W^+ \pm W^-,\mathrm{NS},(1)}(n_f + 1), \\
&H_{i,g}^{W,(1)} = \frac{1}{2}\delta_{i,2} A_{Qg}^{(1)} + C_{i,g}^{W,(1)}(n_f + 1), \\
&H_{3,q}^{W^+ \pm W^-,\mathrm{NS},(1)} = C_{3,q}^{W^+ \pm W^-,\mathrm{NS},(1)}(n_f + 1), \\
&H_{3,g}^{W,(1)} = -\frac{1}{2} A_{Qg}^{(1)},
\end{aligned}
\tag{2.42}
$$

in accordance with the asymptotic expressions derived in Ref. [26]. At 2-loop order, the asymptotic formulae take the form:

$$
\begin{aligned}
&L_{i,q}^{W^+ \pm W^-,\mathrm{NS},(2)} = \delta_{i,2} A_{qq,Q}^{\mathrm{NS},(2)} + C_{i,q}^{W^+ \pm W^-,\mathrm{NS},(2)}(n_f + 1) - C_{i,q}^{W^+ \pm W^-,\mathrm{NS},(2)}(n_f), \\
&H_{i,q}^{W^+ \pm W^-,\mathrm{NS},(2)} = \delta_{i,2} A_{qq,Q}^{\mathrm{NS},(2)} + C_{i,q}^{W^+ \pm W^-,\mathrm{NS},(2)}(n_f + 1), \\
&L_{i,q}^{W,\mathrm{PS},(2)} = C_{i,q}^{W,\mathrm{PS},(2)}(n_f + 1) - C_{i,q}^{W,\mathrm{PS},(2)}(n_f) = 0, \\
&H_{i,q}^{W,\mathrm{PS},(2)} = \frac{1}{2}\delta_{i,2} A_{Qq}^{\mathrm{PS},(2)} + C_{i,q}^{W,\mathrm{PS},(2)}(n_f + 1), \\
&L_{i,g}^{W,(2)} = A_{gg,Q}^{(1)} C_{i,g}^{W,(1)}(n_f + 1) + C_{i,g}^{W,(2)}(n_f + 1) - C_{i,g}^{W,(2)}(n_f), \\
&H_{i,g}^{W,(2)} = A_{gg,Q}^{(1)} C_{i,g}^{W,(1)}(n_f + 1) + C_{i,g}^{W,(2)}(n_f + 1) \\
&\qquad\quad + \frac{1}{2}\Big(\delta_{i,2} A_{Qg}^{(2)} + A_{Qg}^{(1)} C_{i,q}^{W^+ + W^-,\mathrm{NS},(1)}(n_f + 1)\Big), \\
&L_{3,q}^{W^+ \pm W^-,\mathrm{NS},(2)} = A_{qq,Q}^{\mathrm{NS},(2)} + C_{3,q}^{W^+ \pm W^-,\mathrm{NS},(2)}(n_f + 1) - C_{3,q}^{W^+ \pm W^-,\mathrm{NS},(2)}(n_f), \\
&H_{3,q}^{W^+ \pm W^-,\mathrm{NS},(2)} = A_{qq,Q}^{\mathrm{NS},(2)} + C_{3,q}^{W^+ \pm W^-,\mathrm{NS},(2)}(n_f + 1), \\
&H_{3,q}^{W,\mathrm{PS},(2)} = -\frac{1}{2} A_{Qq}^{\mathrm{PS},(2)}, \\
&H_{3,g}^{W,(2)} = \frac{1}{2}\Big(-A_{Qg}^{(2)} - A_{Qg}^{(1)} C_{3,q}^{W^+ + W^-,\mathrm{NS},(1)}(n_f + 1)\Big).
\end{aligned}
\tag{2.43}
$$

Comparing with results given in [23], one finds that the above relations agree for $H_{2,g}^{W,(1)}$ and $H_{2,q}^{W,\mathrm{PS},(2)}$. They further correct $H_{2,g}^{W,(2)}$ with regard to heavy quark loop contributions on external lines, cf. [18], and correct signs in $H_{3,g}^{W,(1)}$, $H_{3,g}^{W,(2)}$ and $H_{3,g}^{W,\mathrm{PS},(2)}$. The correctness of these signs was checked in two ways: First by a careful independent recalculation of the exact 1-loop gluon–boson fusion contributions to $H_{3,g}^{W}$. Then we checked the signs of $H_{3,g}^{W,(1)}$, $H_{3,q}^{W,\mathrm{PS}}$ by calculating their leading $\ln(m^2/Q^2)$-contributions. Details of these calculations are given in Appendix A.

3. The Wilson coefficients

In the following, we present the N-space expressions for the Wilson coefficients having been derived in the previous section. The non-singlet light flavor Wilson coefficients $c_{i,q}^{(i),\mathrm{ns},\pm}$ defined in Eq. (94) of [10] are related to the ones used above via

$$C_{i,q}^{W^+\pm W^-,\mathrm{NS},(i)} = c_{i,q}^{(i),\mathrm{ns},+} \pm c_{i,q}^{(i),\mathrm{ns},-}, \quad i = 2, 3, \tag{3.1}$$

where the $\pm$-signs correspond to each other on the left- and right-hand sides. The splitting denoted by superscripts $+$ or $-$ is the same as in Eq. (14) in [8]. The gluonic and pure singlet Wilson coefficients can be taken over from the electromagnetic case. Using $c_{i,\mathrm{ps}}^{(i)}$ and $c_{i,g}^{(i)}$ from [39], one finds

$$C_{i,q}^{W,\mathrm{PS},(i)}(n_f) = \frac{1}{n_f} c_{i,\mathrm{ps}}^{(i)}, \qquad C_{i,g}^{W,\mathrm{PS},(i)}(n_f) = \frac{1}{n_f} c_{i,g}^{(i)}, \quad i = 2, L. \tag{3.2}$$

The contributions to the non-singlet Wilson coefficients of the structure functions $F_{2,3}$ were given in [6–9], and confirmed in [10].[2]

Very often the massless Wilson coefficients given in the literature for general values of N are understood to be valid either for even or odd values only. The representations for general values of N can, however, be obtained by a Mellin transform of the x-space expressions, cf. e.g. [39,40], resp. for odd moments [41] and the even-/odd-N combinations from [6].

The heavy flavor Wilson coefficients in the Mellin N-space are constructed as described above and are given in terms of harmonic sums [42,43]

$$S_{b,\vec{a}}(N) = \sum_{k=1}^{N} \frac{(\mathrm{sign}(b))^k}{k^{|b|}} S_{\vec{a}}(k), \qquad S_\emptyset = 1, \quad N \in \mathbb{N}\backslash\{0\}. \tag{3.3}$$

For brevity we use the notation $S_{\vec{a}}(N) \equiv S_{\vec{a}}$. The individual Wilson coefficients read:

$$\begin{aligned} L_{2,q}^{W^+ + W^-,\mathrm{NS},(2)} = C_F T_F \Bigg\{ & \frac{16}{3} S_{2,1} - \frac{2(29N^2+29N-6)}{9N(N+1)} {S_1}^2 + \frac{2(35N^2+35N-2)}{3N(N+1)} S_2 \\ & - \frac{2(359N^4+844N^3+443N^2+66N+72)}{27N^2(N+1)^2} S_1 + \frac{P_1}{27N^3(N+1)^3} \\ & - \frac{8}{9} {S_1}^3 + \frac{8}{3} S_2 S_1 - \frac{112 S_3}{9} \Bigg\} + C_F T_F \ln^2\left(\frac{m^2}{Q^2}\right) \end{aligned}$$

[2] See also [41], where also the even–odd-N difference for the Wilson coefficient of F_L is published.

$$\times\left\{\frac{2(3N^2+3N+2)}{3N(N+1)}-\frac{8S_1}{3}\right\}$$
$$+C_FT_F\ln\left(\frac{m^2}{Q^2}\right)\left\{\frac{2(3N^4+6N^3+47N^2+20N-12)}{9N^2(N+1)^2}\right.$$
$$\left.-\frac{80S_1}{9}+\frac{16S_2}{3}\right\}, \tag{3.4}$$

with

$$P_1=795N^6+1587N^5+1295N^4+397N^3+50N^2+300N+216, \tag{3.5}$$

$$H_{2,q}^{W^++W^-,\mathrm{NS},(2)}$$
$$=C_F^2\left\{8(-1)^N\left[5S_{-4}-4S_{-3,1}-2S_{-2,2}+2S_{-3}S_1+4S_{-2,1}S_1+2S_{-2}S_2\right.\right.$$
$$\left.-4S_{-2}{S_1}^2-3\zeta_2S_{-2}+\zeta_2S_2-2\zeta_2{S_1}^2+4\zeta_3S_1-\frac{8}{5}\zeta_2^2\right]+12S_4+40S_{3,1}$$
$$-24S_{2,1,1}-20S_2{S_1}^2+2{S_1}^4-24S_3S_1+16S_{2,1}S_1+24{S_{-2}}^2+6{S_2}^2-8\zeta_2S_2$$
$$+24\zeta_2S_{-2}+16\zeta_2{S_1}^2+16\zeta_3S_1+\frac{64}{5}\zeta_2^2+4(-1)^N\left[-\frac{4(4N-3)}{N(N+1)}S_{-2}S_1\right.$$
$$-\frac{2(4N-3)}{N(N+1)}\zeta_2S_1-\frac{8}{N+1}S_{-3}+\frac{(4N-5)}{N(N+1)}\zeta_3+\frac{8(2N-1)}{N(N+1)}S_{-2,1}$$
$$\left.-\frac{2P_2}{(N-2)N^2(N+1)^2(N+3)}S_{-2}-\frac{P_2}{(N-2)N^2(N+1)^2(N+3)}\zeta_2\right]$$
$$+\frac{2(3N^2+3N-2)}{N(N+1)}{S_1}^3-\frac{2(9N^2+9N-10)}{N(N+1)}S_2S_1$$
$$-\frac{27N^4+26N^3-9N^2-40N-24}{2N^2(N+1)^2}{S_1}^2-\frac{P_3}{2N^3(N+1)^3}S_1+\frac{8(4N-3)}{N(N+1)}\zeta_2S_1$$
$$+\frac{4P_2}{(N-2)N^2(N+1)^2(N+3)}\zeta_2-\frac{4(18N^2-2N+7)}{N(N+1)}\zeta_3$$
$$+\frac{95N^4+162N^3+35N^2-32N-16}{2N^2(N+1)^2}S_2-\frac{2(9N^2+25N-10)}{N(N+1)}S_3$$
$$\left.+\frac{4(3N^2+3N-2)}{N(N+1)}S_{2,1}+\frac{P_4}{8(N-2)N^4(N+1)^4(N+3)}\right\}$$
$$+T_FC_F\ln^2\left(\frac{m^2}{Q^2}\right)\left\{\frac{2(3N^2+3N+2)}{3N(N+1)}-\frac{8S_1}{3}\right\}$$
$$+T_FC_F\ln\left(\frac{m^2}{Q^2}\right)\left\{\frac{2(3N^4+6N^3+47N^2+20N-12)}{9N^2(N+1)^2}-\frac{80S_1}{9}+\frac{16S_2}{3}\right\}$$
$$+T_FC_F\left\{-\frac{8}{9}{S_1}^3-\frac{2(29N^2+29N-6)}{9N(N+1)}{S_1}^2+\frac{2(35N^2+35N-2)}{3N(N+1)}S_2\right.$$
$$-\frac{2(359N^4+844N^3+443N^2+66N+72)}{27N^2(N+1)^2}S_1+\frac{8}{3}S_2S_1+\frac{P_5}{27N^3(N+1)^3}$$

$$
\begin{aligned}
&- \frac{112}{9} S_3 + \frac{16}{3} S_{2,1}\Biggr\} + n_f T_F C_F \Biggl\{ -\frac{8}{9} {S_1}^3 - \frac{2(29N^2+29N-6){S_1}^2}{9N(N+1)} \\
&- \frac{2(247N^4+620N^3+331N^2+66N+72)S_1}{27N^2(N+1)^2} + \frac{8}{3} S_2 S_1 + \frac{P_6}{54N^3(N+1)^3} \\
&+ \frac{2(85N^2+85N-6)S_2}{9N(N+1)} - \frac{88S_3}{9} + \frac{16}{3} S_{2,1}\Biggr\} \\
&+ C_A C_F \Biggl\{ 4(-1)^N \biggl[-5S_{-4} + 4S_{-3,1} + 2S_{-2,2} - 2S_{-3}S_1 - 4S_{-2,1}S_1 - 2S_{-2}S_2 \\
&+ 4S_{-2}{S_1}^2 + 3\zeta_2 S_{-2} - \zeta_2 S_2 + 2\zeta_2 {S_1}^2 - 4\zeta_3 S_1 + \frac{8}{5}\zeta_2^2 \biggr] + \frac{22}{9} {S_1}^3 - 8\zeta_2 {S_1}^2 \\
&+ 4S_2 {S_1}^2 - 32\zeta_3 S_1 + 24S_3 S_1 - 16S_{2,1}S_1 - 12{S_{-2}}^2 - 4{S_2}^2 - 12\zeta_2 S_{-2} + 4\zeta_2 S_2 \\
&- 8S_4 - 24S_{3,1} + 24S_{2,1,1} - \frac{32\zeta_2^2}{5} + \frac{367N^2+367N-66}{18N(N+1)} {S_1}^2 \\
&+ \frac{P_7}{54N^2(N+1)^3} S_1 - \frac{4(4N-3)}{N(N+1)} \zeta_2 S_1 - \frac{2(11N^2+11N+6)}{3N(N+1)} S_2 S_1 \\
&- \frac{2P_2}{(N-2)N^2(N+1)^2(N+3)} \zeta_2 + \frac{2(27N^2+7N+13)}{N(N+1)} \zeta_3 \\
&- \frac{1067N^3+2134N^2+929N-66}{18N(N+1)^2} S_2 + \frac{2(121N^2+193N-72)}{9N(N+1)} S_3 \\
&- \frac{4(11N^2+11N-6)}{3N(N+1)} S_{2,1} - \frac{P_8}{216(N-2)N^3(N+1)^3(N+3)} \\
&+ 2(-1)^N \biggl[-\frac{8(2N-1)}{N(N+1)} S_{-2,1} + \frac{2(4N-3)}{N(N+1)} \zeta_2 S_1 + \frac{4(4N-3)}{N(N+1)} S_{-2} S_1 \\
&+ \frac{P_2}{(N-2)N^2(N+1)^2(N+3)} \zeta_2 - \frac{(4N-5)}{N(N+1)} \zeta_3 + \frac{8}{N+1} S_{-3} \\
&+ \frac{2P_2}{(N-2)N^2(N+1)^2(N+3)} S_{-2} \biggr] \Biggr\},
\end{aligned}
\tag{3.6}
$$

with

$$P_2 = 2N^6 - 2N^5 - 3N^4 + 26N^3 - 45N^2 - 34N - 48, \tag{3.7}$$

$$P_3 = 51N^6 + 203N^5 + 207N^4 + 33N^3 + 106N^2 + 160N + 48, \tag{3.8}$$

$$
\begin{aligned}
P_4 = {} & 331N^{10} + 1179N^9 - 848N^8 - 4754N^7 - 2157N^6 + 4247N^5 + 3474N^4 \\
& - 2528N^3 - 4976N^2 - 2704N - 480,
\end{aligned}
\tag{3.9}
$$

$$P_5 = 795N^6 + 1587N^5 + 1295N^4 + 397N^3 + 50N^2 + 300N + 216, \tag{3.10}$$

$$P_6 = 1371N^6 + 2517N^5 + 1397N^4 + 31N^3 + 140N^2 + 648N + 360, \tag{3.11}$$

$$P_7 = 3155N^5 + 11\,607N^4 + 12\,279N^3 + 3329N^2 + 510N + 792, \tag{3.12}$$

$$
\begin{aligned}
P_8 = {} & 16\,395N^8 + 47\,520N^7 - 51\,416N^6 - 162\,042N^5 - 99\,843N^4 + 7930N^3 \\
& + 21\,432N^2 - 25\,848N - 23\,760,
\end{aligned}
\tag{3.13}
$$

$$
\begin{aligned}
H_{2,q}^{W^+-W^-,\mathrm{NS},(2)} = {} & H_{2,q}^{W^++W^-,\mathrm{NS},(2)} \\
& + C_F(C_F - C_A/2)\bigg\{64(-1)^N S_{-3,1} - \frac{64(-1)^N(2N-1)}{N(N+1)} S_{-2,1} \\
& - 64(-1)^N S_1 S_{-2,1} + 32(-1)^N S_{-2,2} + \frac{16(2N^2+2N+1)}{N^3(N+1)^3} S_1 \\
& - \frac{16 P_9}{(N-2)N^2(N+1)^2(N+2)(N+3)}\zeta_2 \\
& + \frac{16(-1)^N P_{10}}{(N-2)N^2(N+1)^2(N+2)(N+3)} S_{-2} \\
& + \frac{8(-1)^N P_{10}}{(N-2)N^2(N+1)^2(N+2)(N+3)}\zeta_2 \\
& - \frac{4 P_{11}}{(N-2)N^4(N+1)^4(N+2)(N+3)} + 32(-1)^N \zeta_2 {S_1}^2 \\
& + 48(-1)^N \zeta_2 S_{-2} + \frac{16(-1)^N(4N-3)}{N(N+1)} S_1 \zeta_2 - 16(-1)^N \zeta_2 S_2 \\
& - 64(-1)^N \zeta_3 S_1 + 64(-1)^N S_{-2} {S_1}^2 - 80(-1)^N S_{-4} + \frac{64(-1)^N}{N+1} S_{-3} \\
& - 32(-1)^N S_{-3} S_1 + \frac{32(-1)^N(4N-3)}{N(N+1)} S_1 S_{-2} - 32(-1)^N S_{-2} S_2 \\
& + \frac{128}{5}(-1)^N \zeta_2^2 - \frac{8(-1)^N(4N-5)}{N(N+1)}\zeta_3\bigg\},
\end{aligned}
\tag{3.14}
$$

with

$$P_9 = 2N^5 + 6N^4 - 3N^3 - 33N^2 - 26N - 24, \tag{3.15}$$

$$P_{10} = 2N^7 + 2N^6 - 11N^5 + 8N^4 + 13N^3 - 58N^2 - 64N - 48, \tag{3.16}$$

$$P_{11} = N^9 + 6N^8 - 3N^7 + 75N^6 + 278N^5 + 239N^4 - 186N^3 - 386N^2 - 264N - 72, \tag{3.17}$$

$$
\begin{aligned}
H_{2,q}^{W,\mathrm{PS},(2)} = {} & -C_F T_F \ln^2\left(\frac{m^2}{Q^2}\right)\frac{2(N^2+N+2)^2}{(N-1)N^2(N+1)^2(N+2)} \\
& - C_F T_F \ln\left(\frac{m^2}{Q^2}\right)\frac{4(5N^5+32N^4+49N^3+38N^2+28N+8)}{(N-1)N^3(N+1)^3(N+2)^2} \\
& + C_F T_F\bigg\{(-1)^N \frac{32}{(N-1)N(N+1)(N+2)}(2S_{-2}+\zeta_2) \\
& + \frac{4(N^2+N+2)^2}{(N-1)N^2(N+1)^2(N+2)}{S_1}^2 - \frac{8(N^2+N+2)^2}{(N-1)N^2(N+1)^2(N+2)} S_2 \\
& + \frac{8 P_{12}}{(N-1)N^3(N+1)^3(N+2)^2} S_1 + \frac{2 P_{13}}{(N-1)N^4(N+1)^4(N+2)^3} \\
& - \frac{32}{(N-1)N(N+1)(N+2)}\zeta_2\bigg\},
\end{aligned}
\tag{3.18}
$$

with

$$P_{12} = N^7 - 15N^5 - 58N^4 - 92N^3 - 76N^2 - 48N - 16, \tag{3.19}$$

$$P_{13} = 7N^{10} + 36N^9 + 95N^8 + 207N^7 + 583N^6 + 1567N^5 + 2585N^4 + 2464N^3 + 1512N^2 + 656N + 144, \tag{3.20}$$

$$L_{2,g}^{W,(2)} = T_F^2 \ln\left(\frac{m^2}{Q^2}\right)\left\{-\frac{16(N^2+N+2)S_1}{3N(N+1)(N+2)} - \frac{16(N^3-4N^2-N-2)}{3N^2(N+1)(N+2)}\right\}, \tag{3.21}$$

$$\begin{aligned}
H_{2,g}^{W,(2)} = {} & \ln^2\left(\frac{m^2}{Q^2}\right)\left\{-T_F^2\frac{8(N^2+N+2)}{3N(N+1)(N+2)}\right. \\
& + C_F T_F\left[\frac{3N^4+6N^3+11N^2+8N+4}{N^2(N+1)^2(N+2)} - \frac{4(N^2+N+2)}{N(N+1)(N+2)}S_1\right] \\
& \left. + C_A T_F\left[\frac{4(N^2+N+2)}{N(N+1)(N+2)}S_1 - \frac{8(N^4+2N^3+4N^2+3N+2)}{(N-1)N^2(N+1)^2(N+2)^2}\right]\right\} \\
& + \ln\left(\frac{m^2}{Q^2}\right)\left\{T_F^2\left[-\frac{16(N^3-4N^2-N-2)}{3N^2(N+1)(N+2)} - \frac{16(N^2+N+2)S_1}{3N(N+1)(N+2)}\right]\right. \\
& + C_A T_F\left[\frac{4(-1)^N(N^2+N+2)}{N(N+1)(N+2)}(2S_{-2}+\zeta_2) + \frac{4(N^2+N+2)}{N(N+1)(N+2)}S_1{}^2\right. \\
& - \frac{16(2N+3)}{(N+1)^2(N+2)^2}S_1 - \frac{8(N^2+N+2)}{N^3(N+1)(N+2)} \\
& - \frac{4P_{14}}{(N-1)N^3(N+1)^3(N+2)^3} \\
& \left. - \frac{4(N^2+N+2)}{N(N+1)(N+2)}\zeta_2 + \frac{4(N^2+N+2)}{N(N+1)(N+2)}S_2\right] \\
& + C_F T_F\left[-\frac{8(N^2+N+2)}{N(N+1)(N+2)}S_1{}^2 - \frac{2(3N^4+2N^3-9N^2-16N-12)}{N^2(N+1)^2(N+2)}S_1\right. \\
& \left.\left. + \frac{2P_{15}}{N^3(N+1)^3(N+2)} + \frac{8(N^2+N+2)}{N(N+1)(N+2)}S_2\right]\right\} \\
& + C_A T_F\left[-\frac{2(N^2+N+2)}{3N(N+1)(N+2)}S_1{}^3 - \frac{2P_{16}}{(N-1)N(N+1)^2(N+2)^2}S_1{}^2\right. \\
& - \frac{2P_{17}}{(N-1)N^3(N+1)^3(N+2)^3}S_1 - \frac{4(3N^2+3N-2)}{N(N+1)(N+2)}\zeta_2 S_1 \\
& + \frac{26(N^2+N+2)}{N(N+1)(N+2)}S_2 S_1 - \frac{2P_{18}}{(N-1)N^4(N+1)^4(N+2)^4} \\
& - \frac{4P_{19}}{(N-1)N(N+1)^2(N+2)^2}\zeta_2 - \frac{10(N^2+N-6)}{N(N+1)(N+2)}\zeta_3 \\
& + \frac{2P_{20}}{(N-1)N^2(N+1)^2(N+2)^2}S_2 + \frac{8(7N^2+7N+2)}{3N(N+1)(N+2)}S_3 \\
& - \frac{16(N^2+N+2)}{N(N+1)(N+2)}S_{2,1} + 2(-1)^N\left[+\frac{2(3N^2+3N-2)}{N(N+1)(N+2)}\zeta_2 S_1\right.
\end{aligned}$$

$$
\begin{aligned}
&+\frac{4(3N^2+3N-2)}{N(N+1)(N+2)}S_{-2}S_1+\frac{2P_{19}}{(N-1)N(N+1)^2(N+2)^2}\zeta_2\\
&-\frac{(7N^2+7N+6)}{N(N+1)(N+2)}\zeta_3-\frac{2(3N^2+3N+14)}{N(N+1)(N+2)}S_{-3}\\
&+\frac{4P_{19}}{(N-1)N(N+1)^2(N+2)^2}S_{-2}-\frac{4(N^2+N-6)}{N(N+1)(N+2)}S_{-2,1}\Biggr]\Biggr]\\
&+C_FT_F\Biggl[-\frac{22(N^2+N+2)}{3N(N+1)(N+2)}{S_1}^3\\
&-\frac{2(9N^4+9N^3-7N^2-21N-18)}{N^2(N+1)^2(N+2)}{S_1}^2\\
&+\frac{2P_{21}}{N^3(N+1)^3(N+2)}S_1-\frac{64}{N(N+1)(N+2)}\zeta_2S_1\\
&+\frac{10(N^2+N+2)}{N(N+1)(N+2)}S_2S_1+\frac{P_{22}}{(N-2)N^4(N+1)^4(N+2)(N+3)}\\
&-\frac{8P_{23}}{(N-2)N^2(N+1)^2(N+2)(N+3)}\zeta_2+\frac{16(3N^2+3N-4)}{N(N+1)(N+2)}\zeta_3\\
&+\frac{2(10N^3+15N^2+11N-10)}{N^2(N+1)(N+2)}S_2-\frac{8(7N^2+7N-10)}{3N(N+1)(N+2)}S_3\\
&+\frac{16(N^2+N+2)}{N(N+1)(N+2)}S_{2,1}+8(-1)^N\Biggl[+\frac{8}{N(N+1)(N+2)}\zeta_2S_1\\
&+\frac{16}{N(N+1)(N+2)}S_{-2}S_1+\frac{P_{23}}{(N-2)N^2(N+1)^2(N+2)(N+3)}\zeta_2\\
&-\frac{4}{N(N+1)(N+2)}\zeta_3+\frac{8}{N(N+1)(N+2)}S_{-3}-\frac{16}{N(N+1)(N+2)}S_{-2,1}\\
&+\frac{2P_{23}}{(N-2)N^2(N+1)^2(N+2)(N+3)}S_{-2}\Biggr]\Biggr],
\end{aligned}
\tag{3.22}
$$

with

$$
\begin{aligned}
P_{14}&=N^9+6N^8+13N^7+13N^6+8N^5+53N^4+118N^3\\
&\quad+132N^2+104N+32, &(3.23)\\
P_{15}&=4N^6+5N^5-10N^4-39N^3-40N^2-24N-8, &(3.24)\\
P_{16}&=4N^5-7N^4-17N^3-9N^2-57N-10, &(3.25)\\
P_{17}&=15N^9+17N^8-71N^7+81N^6+632N^5+974N^4+984N^3\\
&\quad+664N^2+288N+64, &(3.26)\\
P_{18}&=6N^{12}+48N^{11}+114N^{10}+40N^9-361N^8-1273N^7-3057N^6-5691N^5\\
&\quad-7482N^4-6456N^3-3712N^2-1456N-288, &(3.27)\\
P_{19}&=2N^5-N^4-12N^3+3N^2+32N+24, &(3.28)\\
P_{20}&=4N^6-7N^5-61N^4-49N^3-37N^2-26N-16, &(3.29)\\
P_{21}&=3N^6-14N^5-27N^4-40N^3-74N^2-84N-32, &(3.30)
\end{aligned}
$$

$$P_{22} = 8N^{10} + 28N^9 - 84N^8 - 160N^7 + 465N^6 + 1091N^5 - 163N^4 - 1671N^3 - 1646N^2 - 852N - 216, \tag{3.31}$$

$$P_{23} = N^6 + 7N^5 - 7N^4 - 39N^3 + 14N^2 + 40N + 48, \tag{3.32}$$

$$L_{3,q}^{W^+ + W^-,\mathrm{NS},(2)} = L_{2,q}^{W^+ + W^-,\mathrm{NS},(2)} + C_F T_F \left[\frac{8(2N+1)}{3N(N+1)} S_1 + \frac{4(38N^3 + 27N^2 - 17N - 12)}{9N^2(N+1)^2} \right], \tag{3.33}$$

$$\begin{aligned}
H_{3,q}^{W^+ + W^-,\mathrm{NS},(2)} = H_{2,q}^{W^+ + W^-,\mathrm{NS},(2)} &+ C_F^2 \Bigg\{ \frac{128}{5}(-1)^N \zeta_2^2 + 32(-1)^N {S_1}^2 \zeta_2 \\
&+ \frac{32 P_{24}}{(N-2)(N-1)N^2(N+1)^2(N+2)(N+3)} \zeta_2 \\
&+ \frac{8(-1)^N P_{25}}{(N-2)(N-1)N^2(N+1)^2(N+2)(N+3)} \zeta_2 + 48(-1)^N S_{-2} \zeta_2 \\
&- \frac{16(2N-1)}{N(N+1)} \zeta_2 S_1 + \frac{32(-1)^N (N-1)}{N(N+1)} \zeta_2 S_1 - 16(-1)^N S_2 \zeta_2 \\
&- \frac{4(2N+1)}{N(N+1)} {S_1}^2 + 64(-1)^N S_{-2} {S_1}^2 \\
&+ \frac{P_{26}}{(N-2)(N-1)N^4(N+1)^4(N+2)(N+3)} - \frac{40(2N-1)}{N(N+1)} \zeta_3 \\
&- \frac{16(-1)^N (N-2)}{N(N+1)} \zeta_3 - 80(-1)^N S_{-4} + \frac{16(-1)^N (2N+1)}{N(N+1)} S_{-3} \\
&+ \frac{16(-1)^N P_{25}}{(N-2)(N-1)N^2(N+1)^2(N+2)(N+3)} S_{-2} \\
&+ \frac{2P_{27}}{N^3(N+1)^3} S_1 - 64(-1)^N \zeta_3 S_1 - 32(-1)^N S_{-3} S_1 \\
&+ \frac{64(-1)^N (N-1)}{N(N+1)} S_1 S_{-2} + \frac{4(2N+1)}{N(N+1)} S_2 - 32(-1)^N S_{-2} S_2 \\
&+ \frac{16(2N-1)}{N(N+1)} S_3 + 64(-1)^N S_{-3,1} - \frac{32(-1)^N (2N-1)}{N(N+1)} S_{-2,1} \\
&- 64(-1)^N S_1 S_{-2,1} + 32(-1)^N S_{-2,2} \Bigg\} \\
&+ n_f C_F T_F \left\{ \frac{4(38N^3 + 27N^2 - 17N - 12)}{9N^2(N+1)^2} + \frac{8(2N+1)}{3N(N+1)} S_1 \right\} \\
&+ C_F T_F \left\{ \frac{4(38N^3 + 27N^2 - 17N - 12)}{9N^2(N+1)^2} + \frac{8(2N+1)}{3N(N+1)} S_1 \right\} \\
&+ C_A C_F \Bigg\{ -\frac{64}{5}(-1)^N \zeta_2^2 - 16(-1)^N {S_1}^2 \zeta_2 \\
&- \frac{16 P_{24}}{(N-2)(N-1)N^2(N+1)^2(N+2)(N+3)} \zeta_2 \\
&- \frac{4(-1)^N P_{25}}{(N-2)(N-1)N^2(N+1)^2(N+2)(N+3)} \zeta_2 - 24(-1)^N S_{-2} \zeta_2
\end{aligned}$$

$$
\begin{aligned}
&+\frac{8(2N-1)}{N(N+1)}\zeta_2 S_1-\frac{16(-1)^N(N-1)}{N(N+1)}\zeta_2 S_1 \\
&+8(-1)^N S_2\zeta_2-32(-1)^N S_{-2}S_1{}^2 \\
&+\frac{P_{28}}{9(N-2)(N-1)N^4(N+1)^4(N+2)(N+3)}+\frac{20(2N-1)}{N(N+1)}\zeta_3 \\
&+\frac{8(-1)^N(N-2)}{N(N+1)}\zeta_3+40(-1)^N S_{-4}-\frac{8(-1)^N(2N+1)}{N(N+1)}S_{-3} \\
&-\frac{8(-1)^N P_{25}}{(N-2)(N-1)N^2(N+1)^2(N+2)(N+3)}S_{-2} \\
&-\frac{2(46N^5+67N^4-4N^3-N^2+24N+12)}{3N^3(N+1)^3}S_1+32(-1)^N\zeta_3 S_1 \\
&+16(-1)^N S_{-3}S_1-\frac{32(-1)^N(N-1)}{N(N+1)}S_1S_{-2}+16(-1)^N S_{-2}S_2 \\
&-\frac{8(2N-1)}{N(N+1)}S_3-32(-1)^N S_{-3,1}+\frac{16(-1)^N(2N-1)}{N(N+1)}S_{-2,1} \\
&+32(-1)^N S_1S_{-2,1}-16(-1)^N S_{-2,2}\Bigg\},
\end{aligned}
\tag{3.34}
$$

with

$$P_{24}=N^7+N^6-7N^5-N^4+16N^3-6N^2-4N-12, \tag{3.35}$$

$$P_{25}=2N^8+4N^7-5N^6-N^5-17N^4-67N^3-16N^2+4N+48, \tag{3.36}$$

$$
\begin{aligned}
P_{26}={}&34N^{11}+161N^{10}-135N^9-1238N^8-832N^7+1573N^6+2113N^5 \\
&+1352N^4+884N^3+120N^2-672N-288,
\end{aligned}
\tag{3.37}
$$

$$P_{27}=18N^5+23N^4-4N^3+13N^2+22N+8, \tag{3.38}$$

$$
\begin{aligned}
P_{28}={}&-430N^{11}-2089N^{10}+159N^9+11\,688N^8+11\,736N^7-9189N^6-16\,613N^5 \\
&-8006N^4-3708N^3-1260N^2+2592N+1296,
\end{aligned}
\tag{3.39}
$$

$$
\begin{aligned}
H_{3,q}^{W^+-W^-,\mathrm{NS},(2)}={}&H_{3,q}^{W^++W^-,\mathrm{NS},(2)}+C_F(C_F-C_A/2)\Bigg\{-64(-1)^N S_{-3,1} \\
&+64(-1)^N S_1S_{-2,1}-32(-1)^N S_{-2,2}-\frac{16(2N^2+2N+1)}{N^3(N+1)^3}S_1 \\
&-\frac{16(-1)^N(2N^4+2N^3+N^2+2N-4)}{(N-1)N^2(N+2)}S_{-2} \\
&+\frac{16(N^4+2N^3-3N^2-4N-2)}{(N-1)N^2(N+1)^2(N+2)}\zeta_2 \\
&-\frac{8(-1)^N(2N^4+2N^3+N^2+2N-4)}{(N-1)N^2(N+2)}\zeta_2 \\
&+\frac{4P_{29}}{(N-1)N^4(N+1)^4(N+2)}-32(-1)^N\zeta_2S_1{}^2-48(-1)^N\zeta_2S_{-2} \\
&+\frac{16(-1)^N}{N(N+1)}S_1\zeta_2+16(-1)^N\zeta_2S_2+64(-1)^N\zeta_3S_1
\end{aligned}
$$

$$
\begin{aligned}
&- 64(-1)^N S_{-2} S_1{}^2 + 80(-1)^N S_{-4} - \frac{32(-1)^N}{N(N+1)} S_{-3} \\
&+ 32(-1)^N S_{-3} S_1 + \frac{32(-1)^N}{N(N+1)} S_1 S_{-2} + 32(-1)^N S_{-2} S_2 \\
&- \frac{128}{5}(-1)^N \zeta_2^2 - \frac{24(-1)^N}{N(N+1)} \zeta_3 \Biggr\},
\end{aligned}
\tag{3.40}
$$

with

$$
P_{29} = 9N^8 + 36N^7 + 41N^6 + 13N^5 + 44N^4 + 67N^3 + 20N^2 - 26N - 12, \tag{3.41}
$$

$$
\begin{aligned}
H_{3,q}^{W,\mathrm{PS},(2)} = {} & C_F T_F \ln^2\left(\frac{m^2}{Q^2}\right) \frac{2(N^2+N+2)^2}{(N-1)N^2(N+1)^2(N+2)} \\
&+ C_F T_F \ln\left(\frac{m^2}{Q^2}\right) \frac{4P_{30}}{(N-1)N^3(N+1)^3(N+2)^2} \\
&+ C_F T_F \Biggl\{ \frac{4(N^2+N+2)^2}{(N-1)N^2(N+1)^2(N+2)} S_2 \\
&- \frac{2P_{31}}{(N-1)N^4(N+1)^4(N+2)^3} \Biggr\},
\end{aligned}
\tag{3.42}
$$

with

$$
P_{30} = 5N^5 + 32N^4 + 49N^3 + 38N^2 + 28N + 8, \tag{3.43}
$$

$$
\begin{aligned}
P_{31} = {} & N^{10} + 8N^9 + 29N^8 + 49N^7 - 11N^6 - 131N^5 - 161N^4 - 160N^3 \\
& - 168N^2 - 80N - 16,
\end{aligned}
\tag{3.44}
$$

and

$$
\begin{aligned}
H_{3,g}^{W,(2)} = {} & \ln^2\left(\frac{m^2}{Q^2}\right) \Biggl\{ T_F^2 \frac{8(N^2+N+2)}{3N(N+1)(N+2)} + C_A T_F \Biggl(\frac{8(N^4+2N^3+4N^2+3N+2)}{(N-1)N^2(N+1)^2(N+2)^2} \\
&- \frac{4(N^2+N+2)}{N(N+1)(N+2)} S_1 \Biggr) + C_F T_F \Biggl(\frac{4(N^2+N+2)}{N(N+1)(N+2)} S_1 \\
&- \frac{3N^4+6N^3+11N^2+8N+4}{N^2(N+1)^2(N+2)} \Biggr) \Biggr\} + \ln\left(\frac{m^2}{Q^2}\right) \Biggl\{ C_F T_F \Biggl(\frac{8(N^2+N+2) S_1{}^2}{N(N+1)(N+2)} \\
&+ \frac{2(3N^4+2N^3-9N^2-16N-12) S_1}{N^2(N+1)^2(N+2)} - \frac{2P_{32}}{N^3(N+1)^3(N+2)} \\
&- \frac{8(N^2+N+2)}{N(N+1)(N+2)} S_2 \Biggr) + C_A T_F \Biggl(- \frac{4(N^2+N+2)}{N(N+1)(N+2)} S_1{}^2 \\
&+ \frac{16(2N+3)}{(N+1)^2(N+2)^2} S_1 + \frac{4P_{33}}{(N-1)N^3(N+1)^3(N+2)^3} \\
&- \frac{4(-1)^N(N^2+N+2)}{N(N+1)(N+2)} \zeta_2 + \frac{4(N^2+N+2)}{N(N+1)(N+2)} \zeta_2 \\
&- \frac{8(-1)^N(N^2+N+2)}{N(N+1)(N+2)} S_{-2} - \frac{4(N^2+N+2)}{N(N+1)(N+2)} S_2 \Biggr) \Biggr\}
\end{aligned}
$$

$$
\begin{aligned}
&+C_F T_F\Biggl\{\frac{2(N^2+N+2)}{3N(N+1)(N+2)}{S_1}^3-\frac{2(3N+2)}{N^2(N+2)}{S_1}^2\\
&-\frac{2(N^4-N^3-20N^2-10N-4)}{N^2(N+1)^2(N+2)}S_1+\frac{2(N^2+N+2)}{N(N+1)(N+2)}S_1S_2\\
&-\frac{P_{34}}{N^4(N+1)^4(N+2)}-\frac{2(N^4+17N^3+17N^2-5N-2)}{N^2(N+1)^2(N+2)}S_2\\
&-\frac{8(N^2+N+2)}{3N(N+1)(N+2)}S_3\Biggr\}+C_A T_F\Biggl\{-\frac{2(N^2+N+2)}{3N(N+1)(N+2)}{S_1}^3\\
&+\frac{2(N^3+8N^2+11N+2)}{N(N+1)^2(N+2)^2}{S_1}^2+\frac{2P_{35}}{N(N+1)^3(N+2)^3}S_1\\
&-\frac{4(-1)^N(N^2+N+2)}{N(N+1)(N+2)}S_1\zeta_2+\frac{4(N^2+N+2)}{N(N+1)(N+2)}S_1\zeta_2\\
&-\frac{8(-1)^N(N^2+N+2)}{N(N+1)(N+2)}S_1S_{-2}-\frac{6(N^2+N+2)}{N(N+1)(N+2)}S_1S_2\\
&-\frac{2P_{36}}{(N-1)N^4(N+1)^4(N+2)^4}-\frac{4(-1)^N(N^2-N-4)}{(N+1)^2(N+2)^2}\zeta_2\\
&+\frac{4(N^2-N-4)}{(N+1)^2(N+2)^2}\zeta_2+\frac{2(-1)^N(N^2+N+2)}{N(N+1)(N+2)}\zeta_3-\frac{2(N^2+N+2)}{N(N+1)(N+2)}\zeta_3\\
&-\frac{4(-1)^N(N^2+N+2)}{N(N+1)(N+2)}S_{-3}-\frac{8(-1)^N(N^2-N-4)}{(N+1)^2(N+2)^2}S_{-2}\\
&+\frac{2(7N^5+21N^4+13N^3+21N^2+18N+16)}{(N-1)N^2(N+1)^2(N+2)^2}S_2\\
&-\frac{16(N^2+N+2)}{3N(N+1)(N+2)}S_3+\frac{8(-1)^N(N^2+N+2)}{N(N+1)(N+2)}S_{-2,1}\Biggr\},
\end{aligned}
\tag{3.45}
$$

with

$$P_{32}=4N^6+9N^5-23N^3-26N^2-20N-8, \tag{3.46}$$

$$
\begin{aligned}
P_{33}=\;&N^9+6N^8+15N^7+25N^6+36N^5+85N^4+128N^3\\
&+104N^2+64N+16,
\end{aligned}
\tag{3.47}
$$

$$P_{34}=12N^8+52N^7+132N^6+216N^5+191N^4+54N^3-25N^2-20N-4, \tag{3.48}$$

$$P_{35}=N^6+8N^5+23N^4+54N^3+94N^2+72N+8, \tag{3.49}$$

$$
\begin{aligned}
P_{36}=\;&2N^{12}+20N^{11}+86N^{10}+192N^9+199N^8-N^7-297N^6-495N^5\\
&-514N^4-488N^3-416N^2-176N-32.
\end{aligned}
\tag{3.50}
$$

The harmonic sums appearing are reduced to the following basis:

$$
\begin{aligned}
&\bigl\{S_1(N), S_2(N), S_{-2}(N), S_3(N), S_{-3}(N), S_{2,1}(N), S_{-2,1}(N),\\
&S_4(N), S_{-4}(N), S_{3,1}(N), S_{-3,1}(N), S_{-2,2}(N), S_{2,1,1}(N)\bigr\}.
\end{aligned}
\tag{3.51}
$$

As the harmonic sums, the different Wilson coefficients obey recursion relations for $N \to N-1$, $N \in \mathbb{C}$, which may be used in their analytic continuation. In this way one may shift a value $N \in \mathbb{C}$

to a complex number with large negative real part for which the asymptotic representation of the corresponding Wilson coefficient holds in the analyticity region $N \neq -k$, $k \in \mathbb{N}$. As examples the asymptotic representation for two Wilson coefficients is given in Appendix B.

Since QCD analyses of deep-inelastic scattering data are often being performed in x-space, we present the heavy flavor Wilson coefficients also in this space in Appendix C. The corresponding Mellin inversions were performed using the package HarmonicSums [35,44–46]. Here harmonic polylogarithms [47] occur, which are reduced to the following basis set:

$$\begin{aligned}&\{H_0(x), H_1(x), H_{-1}(x), H_{0,1}(x), H_{0,-1}(x), H_{0,0,1}(x), H_{0,0,-1}(x),\\&H_{0,1,1}(x), H_{0,1,-1}(x), H_{0,-1,1}(x), H_{0,-1,-1}(x)\}.\end{aligned} \tag{3.52}$$

These functions have the following representations in terms of Nielsen integrals [48]:

$$S_{n,p}(x) = \frac{(-1)^{n+p-1}}{(n-1)!p!}\int_0^1 \frac{dz}{z}\ln^{n-1}(z)\ln^p(1-zx), \tag{3.53}$$

$$\mathrm{Li}_n(x) = S_{n-1,1}(x) \tag{3.54}$$

and are given by

$$H_0(x) = \ln(x), \tag{3.55}$$

$$H_1(x) = -\ln(1-x), \tag{3.56}$$

$$H_{-1}(x) = \ln(1+x), \tag{3.57}$$

$$H_{0,1}(x) = \mathrm{Li}_2(x), \tag{3.58}$$

$$H_{0,-1}(x) = \mathrm{Li}_2(-x), \tag{3.59}$$

$$H_{0,0,1}(x) = \mathrm{Li}_3(x), \tag{3.60}$$

$$H_{0,0,-1}(x) = \mathrm{Li}_3(-x), \tag{3.61}$$

$$H_{0,1,1}(x) = S_{1,2}(x), \tag{3.62}$$

$$\begin{aligned}H_{0,-1,1}(x) = {}&\frac{1}{2}\zeta_2\ln(2) - \frac{1}{8}\zeta_3 - \frac{1}{6}\ln^3(2) + \frac{1}{2}\ln(1-x)\zeta_2 + \ln(1-x)\mathrm{Li}_2(-x)\\&- 2\ln(1+x)\zeta_2 + \frac{1}{2}\ln(1+x)\ln^2(2) - \frac{1}{2}\ln^2(1+x)\ln(2) + \frac{1}{6}\ln^3(1+x)\\&+ \ln(x)\ln(1-x)\ln(1+x) - \frac{1}{2}\ln(x)\ln^2(1+x) + \mathrm{Li}_3\left(\frac{1+x}{2}\right)\\&+ \mathrm{Li}_3(1-x) - \mathrm{Li}_3(-x) + \mathrm{Li}_3\left(-\frac{1-x}{1+x}\right) - \mathrm{Li}_3\left(\frac{1-x}{1+x}\right)\\&- \mathrm{Li}_3\left(\frac{2x}{1+x}\right) + \mathrm{Li}_3(x),\end{aligned} \tag{3.63}$$

$$
\begin{aligned}
H_{0,-1,1}(x) = {} & -\frac{1}{2}\zeta_2 \ln(2) - \frac{1}{8}\zeta_3 + \frac{1}{6}\ln^3(2) + \frac{3}{2}\ln(1+x)\zeta_2 - \frac{1}{2}\ln(1+x)\ln^2(2) \\
& + \frac{1}{2}\ln^2(1+x)\ln(2) - \frac{1}{3}\ln^3(1+x) + \ln(1+x)\mathrm{Li}_2(x) \\
& + \frac{1}{2}\ln(x)\ln^2(1+x) - \mathrm{Li}_3\left(\frac{1+x}{2}\right) + \mathrm{Li}_3(-x) + \mathrm{Li}_3\left(\frac{1}{1+x}\right) \\
& + \mathrm{Li}_3\left(\frac{2x}{1+x}\right) - \mathrm{Li}_3(x), \qquad (3.64)
\end{aligned}
$$

$$
H_{0,-1,-1}(x) = S_{1,2}(-x), \qquad (3.65)
$$

see also [10]. Fast numerical implementations for the functions $\mathrm{Li}_{2,3}(x)$ and $S_{1,2}(x)$ are provided in the code ANCONT [49].

4. Numerical results

In the following we illustrate the effect of the heavy flavor Wilson coefficients up to $O(\alpha_s^2)$ on the structure functions $F_i(x, Q^2)$, $i = 1, 2, 3$ for W^+-exchange and the respective differences $F_i^{W^+}(x, Q^2) - F_i^{W^-}(x, Q^2)$ within the kinematic range of HERA referring to the PDFs of Ref. [50]. In Fig. 4 these distributions are given at different values of Q^2 comparing the contributions at LO, NLO, and NNLO. While the difference between the LO and NLO terms are generally large for the individual structure functions due to the newly contributing gluonic term at NLO, the effect is less pronounced in the differences $F_i^{W^+}(x, Q^2) - F_i^{W^-}(x, Q^2)$, showing the typical valence-type shape. In general the NNLO corrections are close to the NLO ones over a wide range in Q^2 displaying the scale evolution of the charged current structure functions. Both the functions $F_i^{W^+}(x, Q^2)$ and $F_i^{W^-}(x, Q^2)$ grow with rising Q^2 and towards small values of x.

Using the expressions derived in the previous section, a FORTRAN program was developed to calculate the 2-loop charm contribution to the structure functions F_2 and F_3. The code is based on earlier work on the exact 1-loop contributions [26]. It works in N-space using the analytic continuation of the N-space representation to complex values of N. The Mellin inversion into the physical x-space is performed using a single complex contour integral picking up the residues of all poles on the real axis. Since the necessary points on the contour can be held fixed for different values of x at a given value of Q^2 for all PDFs, the calculation is naturally very fast. For the analytic continuation of the harmonic sums the ANCONT implementations of Mellin transforms [49,51–53] are used. The numerical accuracy of the implementation is checked by calculating test values of $F_{2,c}$ and $F_{3,c}$ for different values of x. For this purpose we used shape-fits to the ABM11 PDF sets at $Q^2 = 100\ \mathrm{GeV}^2$ given by:

$$
\begin{aligned}
g(x) &= 2.37x^{-0.3}(1-x)^{12}, & s(x) = \bar{s}(x) &= 0.108x^{-0.29}(1-x)^{10}, \\
d(x) &= \left(0.145x^{-0.27} + 1.6x^{0.6}\right)(1-x)^{4.5}, & \bar{d}(x) &= 0.14x^{-0.275}(1-x)^7, \\
u(x) &= \left(0.16x^{-0.26} + 3.5x^{0.7}\right)(1-x)^{3.7}, & \bar{u}(x) &= 0.14x^{-0.275}(1-x)^9. \qquad (4.1)
\end{aligned}
$$

The relative numerical uncertainties of the 2-loop contributions are below 10^{-3} for a wide range of x values. So in total the numeric uncertainties are at the order of 10^{-5} and at least 3 orders of magnitude smaller than the 2-loop corrections, see Fig. 5.

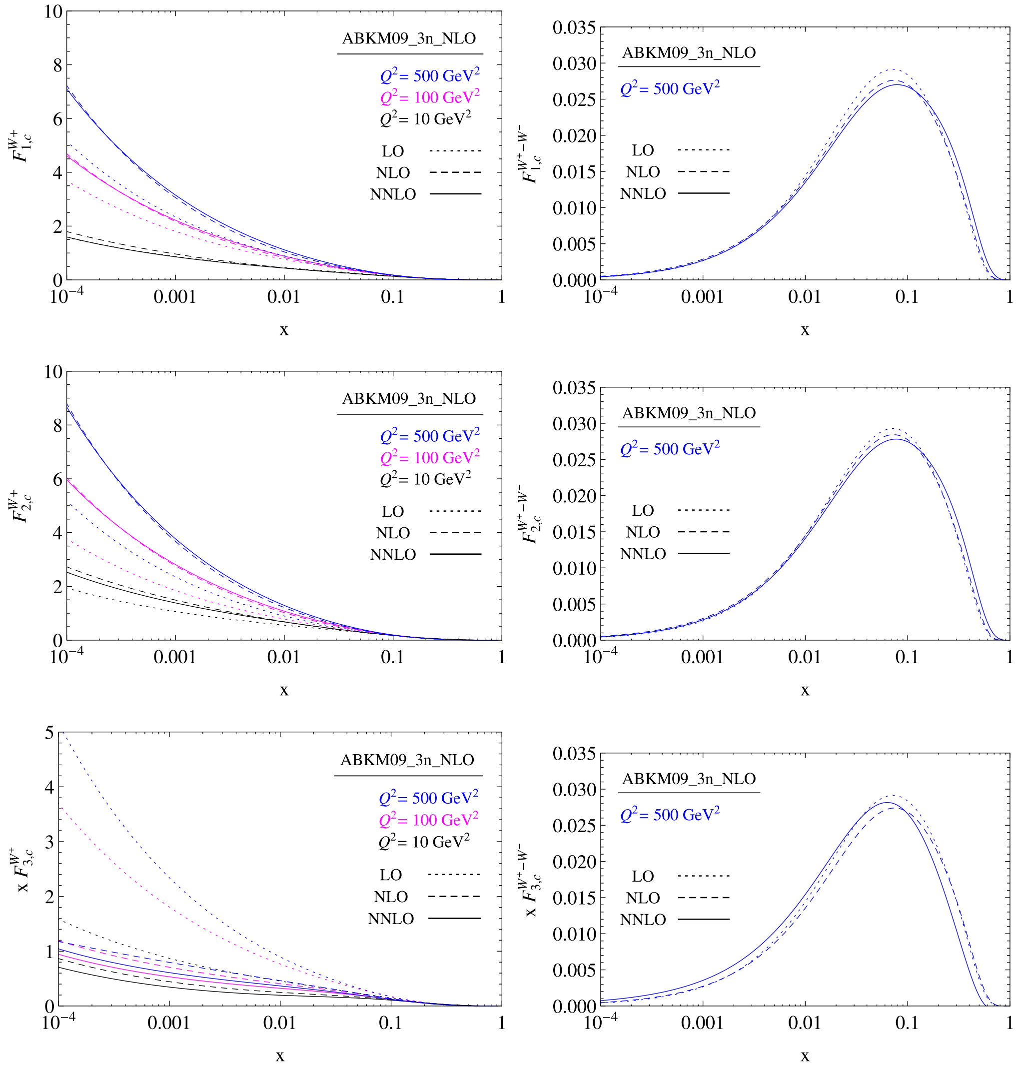

Fig. 4. The charm contributions to the structure functions F_1, F_2, F_3 at different scales Q^2 and with increasing precision: LO, NLO, NNLO using the ABKM09 parameterization [50].

5. Conclusions

The $O(\alpha_s^2)$ QCD corrections to the heavy flavor contributions of the deep-inelastic structure functions in charged-current scattering have been calculated in the region $Q^2 \gg m^2$ using the method of Ref. [4]. We completed the set of Wilson coefficients and corrected previous results in Ref. [23], presenting a detailed outline of the differences found. The Wilson coefficients obey representations in terms of harmonic sums in the Mellin-N space and weighted harmonic polylogarithms in x-space, respectively, in both cases to weight $\mathsf{w} = 4$. Numerical studies were performed for the structure functions $F_{i,c}^{W^\pm}(x, Q^2)$, $i = 1, 2, 3$ in the kinematic region available at HERA comparing the corrections from LO to NNLO. The NNLO results come out close to those

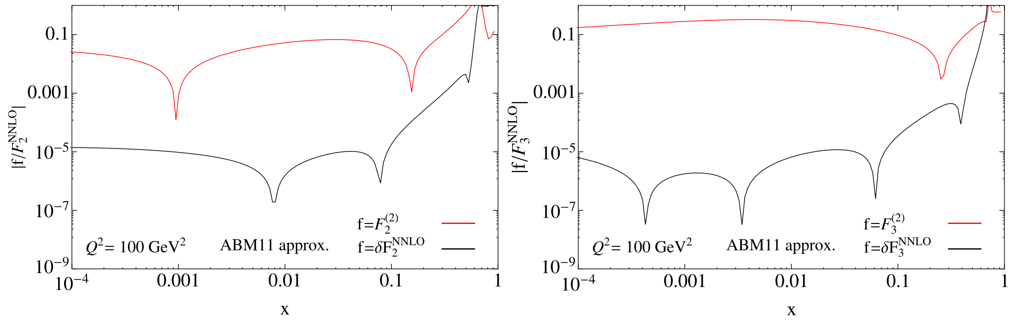

Fig. 5. The relative numerical precision of the NNLO heavy quark contributions structure functions for charm production (black) in comparison with the ratio of the $O(\alpha_s^2)$ correction to the structure function (red). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

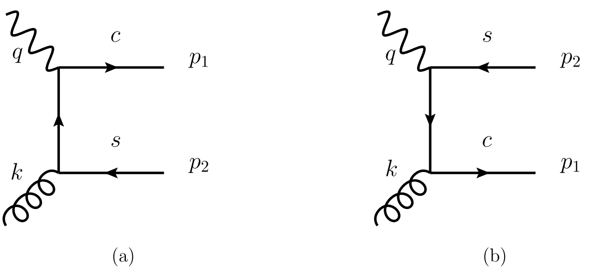

Fig. 6. Graphs contributing to $H_g^{(2)}$.

at NLO in a wide range of Q^2. Numerical implementations both in the Mellin-N and x-space were performed at high accuracy. The Wilson coefficients in x-space may all be expressed in terms of Nielsen integrals. The corresponding FORTRAN codes are available on request.

Acknowledgements

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Appendix A. Relative signs in Wilson coefficients

Since the sign in front of the OME in the gluonic heavy flavor Wilson coefficient of Eq. (2.38) contradicts the asymptotic representation given in Eq. (A.17) of [23], a recalculation of the full gluonic $O(\alpha_s)$ correction was performed which will be presented in the following. We confirm the result given in [25,54]. As the minus sign was confirmed in this analysis, further changes in signs in the relations (A.18) and (A.19) of [23] are anticipated. The reasoning follows the idea of calculating leading logarithms in the Altarelli–Parisi picture of scaling violations [55].

The heavy flavor Wilson coefficient H_g is obtained from the diagrams in Fig. 6 with the matrix element

$$M_a^\mu = \bar{u}(p_1) i\gamma^\mu(1-\gamma_5)\frac{i(\not{p}_1-\not{q})}{(p_1-q)^2}\gamma^\rho i g_s t^a v(p_2)\varepsilon_\rho^a(k)$$
$$+\bar{u}(p_1)\gamma^\rho i g_s t^a \frac{i(\not{p}_1-\not{k}+m)}{(p_1-k)^2-m^2} i\gamma^\mu(1-\gamma_5)v(p_2)\varepsilon_\rho^a(k) \tag{A.1}$$

contributing to the hadronic tensor. For the implementation of γ_5, the prescription of [56] was used, which amounts to the replacement

$$\gamma_\mu\gamma_5 = \frac{i}{6}\varepsilon_{\mu\nu\rho\sigma}\gamma_\nu\gamma_\rho\gamma_\sigma, \tag{A.2}$$

in the matrix element, where products of Levi-Civita symbols are evaluated by the determinant

$$\varepsilon_{\alpha\beta\gamma\delta}\varepsilon_{\mu\nu\rho\sigma} = \begin{vmatrix} g_{\alpha\mu} & g_{\alpha\nu} & g_{\alpha\rho} & g_{\alpha\sigma} \\ g_{\beta\mu} & g_{\beta\nu} & g_{\beta\rho} & g_{\beta\sigma} \\ g_{\gamma\mu} & g_{\gamma\nu} & g_{\gamma\rho} & g_{\gamma\sigma} \\ g_{\delta\mu} & g_{\delta\nu} & g_{\delta\rho} & g_{\delta\sigma} \end{vmatrix}, \tag{A.3}$$

and Lorentz contractions are performed in D dimensions. Since $O(\alpha_s)$ is the leading order of the gluon channel, no finite renormalization is needed. The Lorentz-structure of the squared matrix element is projected onto the (unrenormalized) partonic versions of the structure functions $\hat{\mathcal{F}}_i$, $i = 1, 2, 3$ via the projectors:

$$\hat{P}_1 = \frac{1}{2+\varepsilon}\frac{x}{Q^2}\left(4xP_\mu P_\nu + 2P_\mu q_\nu + 2P_\nu q_\mu - \frac{Q^2}{x}g_{\mu\nu}\right),$$
$$\hat{P}_2 = 2x\left(\frac{q_\mu q_\nu}{Q^2} - \frac{g_{\mu\nu}}{2+\varepsilon}\right) + 4\frac{x^2}{Q^2}\frac{3+\varepsilon}{2+\varepsilon}(2xP_\mu P_\nu + P_\mu q_\nu + P_\nu q_\mu),$$
$$\hat{P}_3 = -\frac{4x}{Q^2}\frac{1}{(1+\varepsilon)(2+\varepsilon)} i\varepsilon_{\mu\nu\rho\sigma}P^\rho q^\sigma. \tag{A.4}$$

The two particle phase space leads to one-dimensional integrals which, after a partial fraction decomposition, can be solved in terms of ${}_2F_1$ functions, e.g.

$$\int_0^1 dy\, y^{\frac{\varepsilon}{2}}(1-y)^{\frac{\varepsilon}{2}}\frac{1}{(p_1-k)^2-m^2}$$
$$= -\frac{1}{s+Q^2}B\left(1+\frac{\varepsilon}{2}, 1+\frac{\varepsilon}{2}\right){}_2F_1\left[\begin{matrix}1, 1+\frac{\varepsilon}{2}\\ 2+\varepsilon\end{matrix}; \frac{s-m^2}{s}\right], \tag{A.5}$$

with

$$y = \frac{1}{2}\left[1+\cos\sphericalangle(p_1,q)\right], \quad (p_1-k)^2-m^2 = -\left(s+Q^2\right)\left(1-\frac{s-m^2}{s}(1-y)\right). \tag{A.6}$$

This particular example is the source for the mass logarithms:

$${}_2F_1\left[\begin{matrix}1, 1+\frac{\varepsilon}{2}\\ 2+\varepsilon\end{matrix}; \frac{s-m^2}{s}\right] = -\frac{s}{s-m^2}\ln\left(\frac{m^2}{s}\right) + O(\varepsilon), \tag{A.7}$$

and thus contributes to the OME in the asymptotic expansions.

The t-channel exchange of the light s-quark in the first diagram of Fig. 6 introduces a collinear singularity, which has to be removed via mass factorization as described in Eq. (2.38) of [9]. In the present case it proceeds via:

$$\hat{\mathcal{F}}_i = \Gamma_{qg}^{(1)} + H_{i,g}^{(1)}, \tag{A.8}$$

with the $\overline{\text{MS}}$ transition function

$$\Gamma_{qg}^{(1)} = S_\varepsilon \frac{1}{2\varepsilon} P_{qg}^{(0)}, \quad P_{qg}^{(0)}(z) = 8T_F\left[z^2 + (1-z)^2\right]. \tag{A.9}$$

In contrast to the electromagnetic case, the factor $2n_f$ in (2.38) of [9] is omitted, since the above calculation is performed for only one incoming light flavor, and only for one of the two graphs in Fig. 6 the quark propagator is massless and thus develops a collinear singularity. The results of this calculation agree with those in [24,25,54].

In order to gain further confidence in the emergence of a minus sign in the asymptotic representation, as well as to understand how this observation relates to the pure singlet Wilson coefficients at 2-loop order, the calculation of the leading logarithmic contributions is performed using the method also applied by Altarelli and Parisi [55], cf. also [57].

A Sudakov parameterization [58] is introduced for the t-channel momentum in the diagram in Fig. 6(a):

$$k - p_2 = \alpha k + \beta q' + k_\perp, \tag{A.10}$$

denoting the gluon momentum by k, and the photon momentum by q. Furthermore, the vectors $k_\perp$ and q' are defined via

$$q' = q + xk, \qquad q'.k_\perp = k.k_\perp = 0. \tag{A.11}$$

This leads to the final state momenta

$$p_1 = (\alpha - x)k + (\beta + 1)q' + k_\perp, \tag{A.12}$$
$$p_2 = (1-\alpha)k - \beta q' - k_\perp, \tag{A.13}$$

and the Mandelstam variables

$$\begin{aligned} s &:= (q+k)^2 = 2k.q - Q^2, \\ t &:= (p_1 - q)^2, \\ u &:= (p_1 - k)^2 = -t + m^2 - Q^2 - s. \end{aligned} \tag{A.14}$$

With the approximation $q'^2 \approx k^2 \approx 0$ and $p.q' \approx p.q$, the phase space integral then takes the form

$$\begin{aligned} \int dp_1\, dp_2\, \delta\left(p_1^2 - m^2\right)\delta\left(p_2^2\right) = \int d\beta\, d\alpha\, dk_\perp^2 \frac{\pi}{2k.q(1-\alpha)} \delta\left(\beta - \frac{k_\perp^2}{2k.q(1-\alpha)}\right) \\ \times \delta\left(\alpha - x + \frac{1-x}{1-\alpha}\frac{k_\perp^2}{2k.q} - \frac{m^2}{2k.q}\right). \end{aligned} \tag{A.15}$$

Using the implication from the δ-distributions one finds

$$k_\perp^2 = (1-\alpha)t, \tag{A.16}$$

and thus defines the positive variable

$$r^2 := -t. \tag{A.17}$$

The physical region[3] is determined from the conditions

$$0 \leqslant \cos \sphericalangle(q, p_1) \leqslant 1 \quad \text{and} \quad 0 \leqslant \cos \sphericalangle(q, p_2) \leqslant 1 \tag{A.18}$$

on the angles in the target system of coordinates. As a result one finds

$$2k^2 \frac{s-m^2}{s+Q^2} \leqslant r^2 \leqslant \frac{(s-m^2)(s+Q^2)}{s}. \tag{A.19}$$

There are two integrals leading to logarithmic values:

$$\begin{aligned}\int dr^2 \frac{1}{(p_1-q)^2} &= -\int dr^2 \frac{1}{r^2} \\ &= \ln\left(\frac{2sk^2}{(s+Q^2)^2}\right) \approx \ln\left(\frac{k^2}{s}\right),\end{aligned} \tag{A.20}$$

$$\begin{aligned}\int dr^2 \frac{1}{(p_1-k)^2-m^2} &= \int dr^2 \frac{1}{r^2-s-Q^2} \\ &= -\ln\left(\frac{s}{m^2} - 2\frac{k^2}{m^2}\frac{s}{s+Q^2}\frac{(s-m^2)}{s+Q^2}\right) \\ &\approx \ln\left(\frac{m^2}{s}\right).\end{aligned} \tag{A.21}$$

Since the incoming gluon is massless, i.e. $k^2 = 0$, the first logarithm represents a collinear singularity, which was earlier regulated in $D = 4 + \varepsilon$ dimensions and removed via mass factorization in Eq. (A.8). The second logarithm indeed constitutes the leading mass dependence of the process. Picking out this logarithmic part, one finds

$$H_{g,1}^{W,(1),\mathrm{LL}} = H_{g,2}^{W,(1),\mathrm{LL}} = -\frac{1}{2} P_{qg}^{(0)}(N) \ln\left(\frac{m^2}{Q^2}\right), \tag{A.22}$$

$$H_{g,3}^{W,(1),\mathrm{LL}} = \frac{1}{2} P_{qg}^{(0)}(N) \ln\left(\frac{m^2}{Q^2}\right). \tag{A.23}$$

The splitting functions derive from the fermion traces after applying the above approximations and canceling against denominators.

In order to obtain the 2-loop pure singlet contribution in leading logarithmic approximation, one has to include another ladder rung formed by a light quark line, as depicted in Fig. 7.

Then the Sudakov parameters are introduced as above:

$$k_1 = \alpha_1 k + \beta_1 q' + k_{\perp 1}, \tag{A.24}$$

$$k_2 = \alpha_2 k + \beta_2 q' + k_{\perp 2}. \tag{A.25}$$

The three-particle phase space can be treated similarly as before, assuming a strict hierarchy $k^2 \ll |k_{\perp 1}^2| \ll |k_{\perp 2}^2| \ll Q^2$. The δ-distributions introduced by the phase space integral then take the forms:

[3] For a collection of kinematic formulae used here see [59].

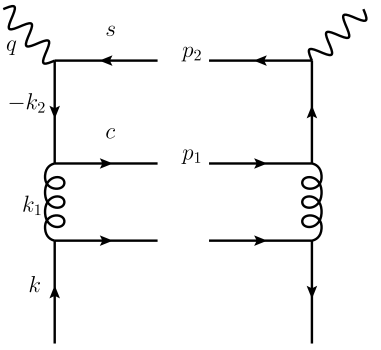

Fig. 7. The leading logarithmic 2-loop PS-contribution $H_q^{\mathrm{PS},(2),\mathrm{LL}}$ can be built from the leading logarithmic 1-loop gluonic contribution by adding a splitting of a quark into a gluon.

$$\begin{aligned}
\delta\big((k-k_1)^2\big) &= \frac{1}{2k.q(1-\alpha_1)}\delta\left(\beta_1 - \frac{k_{\perp 1}^2}{2k.q(1-\alpha_1)}\right),\\
\delta\big((k_2-k_1)^2-m^2\big) &= \frac{1}{2k.q(\alpha_1-\alpha_2)}\delta\left(\beta_2 - \frac{k_{\perp 2}^2-m^2}{2k.q(\alpha_1-\alpha_2)}\right),\\
\delta\big((k_2+q)^2\big) &= \frac{1}{2k.q}\delta\left(\alpha_2 - x + \frac{(\alpha_1-x)}{(\alpha_1-\alpha_2)}\frac{(k_\perp^2-m^2)}{2k.q}\right).
\end{aligned} \tag{A.26}$$

This again leads to the definition of positive squares of momenta:

$$\begin{aligned}
r_1^2 &= -\frac{k_{\perp 1}^2}{1-\alpha_1},\\
r_2^2 &= -\frac{k_{\perp 2}^2}{\alpha_1-\alpha_2}.
\end{aligned} \tag{A.27}$$

Like in the case of purely massless ladder rungs [55], see also [57,60], the integral becomes nested in both the momentum and the Sudakov variables α_1, α_2,

$$\begin{aligned}
H_{3,q}^{W,\mathrm{PS},(2)} = \frac{1}{8} &\int\limits_{k^2}^{\frac{(s-m^2)(s+Q^2)}{s}} \frac{dr_2^2}{r_2^2-s-Q^2} \int\limits_{k^2}^{|k_{\perp 2}^2|} \frac{d|k_{\perp 1}^2|}{-|k_{\perp 1}^2|}\\
&\times \int\limits_0^1 \frac{d\alpha_2}{\alpha_2}\delta\left(1-\frac{x}{\alpha_2}\right)\int\limits_{\alpha_2}^1 \frac{d\alpha_1}{\alpha_1}P_{gq}^{(0)}\left(\frac{\alpha_1}{\alpha_2}\right)P_{qg}^{(0)}(\alpha_1),
\end{aligned} \tag{A.28}$$

where the following splitting function occurs:

$$P_{gq}^{(0)}(x) = 4C_F\frac{1+(1-x)^2}{x}. \tag{A.29}$$

With the variable substitution $R^2 = s + Q^2 - r_2^2$, the integrals over the squared momenta can be performed:

$$\int_{k^2}^{\frac{(s-m^2)(s+Q^2)}{s}} \frac{dr_2^2}{r_2^2 - s - Q^2} \int_{k^2}^{|k_{\perp 2}^2|} \frac{d|k_{\perp 1}^2|}{-|k_{\perp 1}^2|} = \int_{m^2\frac{s+Q^2}{s}}^{s+Q^2-k^2} \frac{dR^2}{R^2} \int_{k^2}^{R^2\frac{\alpha_1-\alpha_2}{\alpha_1}-m^2} \frac{d|k_{\perp 1}^2|}{-|k_{\perp 1}^2|}$$
$$\approx \int_{m^2\frac{s+Q^2}{s}}^{s+Q^2} \frac{dR^2}{R^2} \ln\left(\frac{R^2}{k^2}\frac{\alpha_1-\alpha_2}{\alpha_1}\right)$$
$$= \frac{1}{2}\ln^2\left(\frac{m^2}{Q^2}\right) + O\left(\ln\left(\frac{m^2}{Q^2}\right)\right). \tag{A.30}$$

Here the reference scale in the mass-logarithm was chosen to be Q^2. In the Mellin space the convolutions of the splitting functions in (A.28) factorize, and one finds to $O(\ln^2(m^2/Q^2))$ the relation

$$H_{3,q}^{W,\mathrm{PS},(2)} = \frac{1}{16} P_{qg}^{(0)}(N) P_{gq}^{(0)}(N) \ln^2\left(\frac{m^2}{Q^2}\right) = -\frac{1}{2} A_{Qq}^{\mathrm{PS},(2)}, \tag{A.31}$$

which fixes the respective sign. The additional ladder rung has the effect of introducing another splitting function independently from the boson–quark coupling. Hence the minus sign from the one-loop heavy flavor Wilson coefficient in leading logarithmic approximation is simply translated to the 2-loop pure-singlet contribution. As in the gluonic heavy flavor Wilson coefficient at the 1-loop order, the result above disagrees with the asymptotic representation given in [23]. This confirms the results of the derivation of the asymptotic representations at 2-loop order given in Section 2, which captures the signs in a rigorous way.

Appendix B. Asymptotic expansion of the Wilson coefficients

In this appendix we present the asymptotic expansions of the different Wilson coefficients. In Mellin-space codes these expressions may serve as numerical starting values for large $N \in \mathbb{C}$ outside the singularities being located at the integers left of an integer N_0. All other values in the analytic region can be obtained by the shift relations of the analytic continuations of the harmonic sums, cf. [49,51–53].

As examples we show the asymptotic expansions of $L_{2,q}^{W^++W^-,\mathrm{NS},(2)}$ and $H_{2,q}^{W^++W^-,\mathrm{NS},(2)}$. One obtains

$$\begin{aligned} L_{2,q}^{W^++W^-,\mathrm{NS},(2)} = C_F T_F \Bigg\{ \Bigg[& -\frac{80}{9}\ln(\bar{N}) + \frac{16}{3}\zeta_2 + \frac{2}{3} - \frac{88}{9N} + \frac{356}{27N^2} - \frac{16}{N^3} + \frac{478}{27N^4} \\ & - \frac{304}{15N^5} + \frac{13\,124}{567N^6} - \frac{544}{21N^7} + \frac{767}{27N^8}\Bigg]\ln\left(\frac{m^2}{Q^2}\right) + \Bigg[-\frac{8\ln(\bar{N})}{3} + 2 \\ & - \frac{4}{3N} + \frac{14}{9N^2} - \frac{4}{3N^3} + \frac{59}{45N^4} - \frac{4}{3N^5} + \frac{254}{189N^6} - \frac{4}{3N^7} \\ & + \frac{119}{90N^8}\Bigg]\ln^2\left(\frac{m^2}{Q^2}\right) + \Bigg[-\frac{58}{9} - \frac{4}{3N} + \frac{14}{9N^2} - \frac{4}{3N^3} + \frac{59}{45N^4} \\ & - \frac{4}{3N^5} + \frac{254}{189N^6} - \frac{4}{3N^7} + \frac{119}{90N^8}\Bigg]\ln(\bar{N})^2 + \Bigg[-\frac{718}{27} - \frac{214}{9N} \end{aligned}$$

$$
\begin{aligned}
&+\frac{455}{27N^2}-\frac{182}{9N^3}+\frac{2893}{135N^4}-\frac{1079}{45N^5}+\frac{152\,897}{5670N^6}-\frac{28\,121}{945N^7}\\
&+\frac{521\,287}{16\,200N^8}\bigg]\ln(\bar{N})-\frac{8}{9}\ln(\bar{N})^3+\frac{8}{3}\zeta_2\ln(\bar{N})+\bigg[\frac{70}{3}+\frac{4}{3N}\\
&-\frac{14}{9N^2}+\frac{4}{3N^3}-\frac{59}{45N^4}+\frac{4}{3N^5}-\frac{254}{189N^6}+\frac{4}{3N^7}-\frac{119}{90N^8}\bigg]\zeta_2\\
&-\frac{16\zeta_3}{9}+\frac{265}{9}-\frac{1931}{27N}+\frac{4816}{81N^2}-\frac{139}{2N^3}+\frac{285\,637}{3240N^4}\\
&-\frac{72\,146}{675N^5}+\frac{371\,921}{2835N^6}-\frac{6\,318\,899}{39\,690N^7}+\frac{257\,172\,521}{1\,360\,800N^8}\bigg\}\\
&+O\left(\ln^2(\bar{N})\frac{1}{N^9}\right), \qquad \text{(B.1)}
\end{aligned}
$$

where $\bar{N}=N\exp(\gamma_E)$ and γ_E denotes the Euler–Mascheroni number. The asymptotic representation for $H_{2,q}^{W^++W^-,\mathrm{NS},(2)}$ reads:

$$
\begin{aligned}
H_{2,q}^{W^++W^-,\mathrm{NS},(2)}={}&C_F^2\bigg\{2\ln(\bar{N})^4+\bigg[6+\frac{4}{N}-\frac{14}{3N^2}+\frac{4}{N^3}-\frac{59}{15N^4}+\frac{4}{N^5}-\frac{254}{63N^6}\\
&+\frac{4}{N^7}-\frac{119}{30N^8}\bigg]\ln(\bar{N})^3+\bigg[-\frac{27}{2}+\frac{39}{N}-\frac{33}{2N^2}+\frac{65}{3N^3}-\frac{683}{30N^4}\\
&+\frac{797}{30N^5}-\frac{13\,099}{420N^6}+\frac{1063}{30N^7}-\frac{140\,287}{3600N^8}\bigg]\ln(\bar{N})^2-4\zeta_2\ln(\bar{N})^2\\
&+\bigg[-\frac{51}{2}-\frac{49}{2N}+\frac{287}{4N^2}-\frac{1561}{18N^3}+\frac{332}{5N^4}-\frac{1353}{25N^5}-\frac{32\,719}{3780N^6}\\
&+\frac{533\,443}{4410N^7}-\frac{642\,431}{7200N^8}\bigg]\ln(\bar{N})+\bigg[-18+\frac{28}{N}-\frac{106}{3N^2}+\frac{36}{N^3}\\
&-\frac{541}{15N^4}+\frac{36}{N^5}-\frac{2266}{63N^6}+\frac{36}{N^7}-\frac{1081}{30N^8}\bigg]\zeta_2\ln(\bar{N})+24\zeta_3\ln(\bar{N})\\
&+\frac{4\zeta_2^2}{5}+\bigg(\frac{111}{2}-\frac{47}{N}+\frac{213}{2N^2}-\frac{233}{N^3}+\frac{24\,067}{45N^4}-\frac{42\,331}{30N^5}\\
&+\frac{4\,665\,499}{1260N^6}-\frac{6\,8459\,71}{630N^7}+\frac{2\,391\,890\,629}{75\,600N^8}\bigg)\zeta_2+\bigg(-66+\frac{60}{N}\\
&-\frac{74}{N^2}+\frac{72}{N^3}-\frac{359}{5N^4}+\frac{72}{N^5}-\frac{1514}{21N^6}+\frac{72}{N^7}-\frac{719}{10N^8}\bigg)\zeta_3+\frac{331}{8}\\
&-\frac{431}{4N}+\frac{42}{N^2}+\frac{12\,047}{72N^3}-\frac{864\,961}{1440N^4}+\frac{26\,702\,713}{13\,500N^5}-\frac{144\,440\,651}{25\,200N^6}\\
&+\frac{16\,295\,446\,411}{926\,100N^7}-\frac{82\,008\,282\,247}{1\,587\,600N^8}\bigg\}+C_FT_F\bigg\{-\frac{8}{9}\ln(\bar{N})^3\\
&+\bigg[-\frac{58}{9}-\frac{4}{3N}+\frac{14}{9N^2}-\frac{4}{3N^3}+\frac{59}{45N^4}-\frac{4}{3N^5}+\frac{254}{189N^6}\\
&-\frac{4}{3N^7}+\frac{119}{90N^8}\bigg]\ln(\bar{N})^2+\bigg[-\frac{718}{27}-\frac{214}{9N}+\frac{455}{27N^2}-\frac{182}{9N^3}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2893}{135N^4}-\frac{1079}{45N^5}+\frac{152\,897}{5670N^6}-\frac{28\,121}{945N^7}+\frac{521\,287}{16\,200N^8}\biggr]\ln(\bar{N})\\
&+\ln^2\left(\frac{m^2}{Q^2}\right)\biggl[-\frac{8\ln(\bar{N})}{3}+2-\frac{4}{3N}+\frac{14}{9N^2}-\frac{4}{3N^3}+\frac{59}{45N^4}\\
&-\frac{4}{3N^5}+\frac{254}{189N^6}-\frac{4}{3N^7}+\frac{119}{90N^8}\biggr]+\ln\left(\frac{m^2}{Q^2}\right)\biggl[-\frac{80\ln(\bar{N})}{9}\\
&+\frac{16\zeta_2}{3}+\frac{2}{3}-\frac{88}{9N}+\frac{356}{27N^2}-\frac{16}{N^3}+\frac{478}{27N^4}-\frac{304}{15N^5}+\frac{13\,124}{567N^6}\\
&-\frac{544}{21N^7}+\frac{767}{27N^8}\biggr]+\biggl[\frac{70}{3}+\frac{4}{3N}-\frac{14}{9N^2}+\frac{4}{3N^3}-\frac{59}{45N^4}+\frac{4}{3N^5}\\
&-\frac{254}{189N^6}+\frac{4}{3N^7}-\frac{119}{90N^8}\biggr]\zeta_2+\frac{8}{3}\zeta_2\ln(\bar{N})-\frac{16\zeta_3}{9}+\frac{265}{9}\\
&-\frac{1931}{27N}+\frac{4816}{81N^2}-\frac{139}{2N^3}+\frac{285\,637}{3240N^4}-\frac{72\,146}{675N^5}+\frac{371\,921}{2835N^6}\\
&-\frac{6\,318\,899}{39\,690N^7}+\frac{257\,172\,521}{1\,360\,800N^8}\biggr\}+n_fC_FT_F\biggl\{-\frac{8}{9}\ln(\bar{N})^3+\biggl[-\frac{58}{9}\\
&-\frac{4}{3N}+\frac{14}{9N^2}-\frac{4}{3N^3}+\frac{59}{45N^4}-\frac{4}{3N^5}+\frac{254}{189N^6}-\frac{4}{3N^7}\\
&+\frac{119}{90N^8}\biggr]\ln(\bar{N})^2+\biggl[-\frac{494}{27}-\frac{214}{9N}+\frac{455}{27N^2}-\frac{182}{9N^3}+\frac{2893}{135N^4}\\
&-\frac{1079}{45N^5}+\frac{152\,897}{5670N^6}-\frac{28\,121}{945N^7}+\frac{521\,287}{16\,200N^8}\biggr]\ln(\bar{N})+\frac{8}{3}\zeta_2\ln(\bar{N})\\
&+\biggl[\frac{170}{9}+\frac{4}{3N}-\frac{14}{9N^2}+\frac{4}{3N^3}-\frac{59}{45N^4}+\frac{4}{3N^5}-\frac{254}{189N^6}+\frac{4}{3N^7}\\
&-\frac{119}{90N^8}\biggr]\zeta_2+\frac{8\zeta_3}{9}+\frac{457}{18}-\frac{1699}{27N}+\frac{3668}{81N^2}-\frac{859}{18N^3}+\frac{21\,229}{360N^4}\\
&-\frac{46\,946}{675N^5}+\frac{711\,653}{8505N^6}-\frac{3\,962\,699}{39\,690N^7}+\frac{158\,737\,961}{1\,360\,800N^8}\biggr\}\\
&+C_FC_A\biggl(\frac{22\ln(\bar{N})^3}{9}+\biggl(\frac{367}{18}+\frac{11}{3N}-\frac{77}{18N^2}+\frac{11}{3N^3}-\frac{649}{180N^4}\\
&+\frac{11}{3N^5}-\frac{1397}{378N^6}+\frac{11}{3N^7}-\frac{1309}{360N^8}\biggr)\ln(\bar{N})^2-4\zeta_2\ln(\bar{N})^2\\
&+\biggl(\frac{3155}{54}+\frac{1333}{18N}-\frac{8365}{108N^2}+\frac{161}{2N^3}-\frac{44\,219}{540N^4}+\frac{75\,433}{900N^5}\\
&-\frac{1\,451\,011}{22\,680N^6}+\frac{531\,973}{26\,460N^7}-\frac{21\,246\,299}{453\,600N^8}\biggr)\ln(\bar{N})\\
&+\biggl(-\frac{22}{3}-\frac{20}{N}+\frac{74}{3N^2}-\frac{24}{N^3}+\frac{359}{15N^4}-\frac{24}{N^5}+\frac{1514}{63N^6}-\frac{24}{N^7}\\
&+\frac{719}{30N^8}\biggr)\zeta_2\ln(\bar{N})-40\zeta_3\ln(\bar{N})+\frac{51}{5}\zeta_2^2+\biggl(-\frac{1139}{18}+\frac{13}{3N}
\end{aligned}
$$

$$
-\frac{805}{18N^2}+\frac{326}{3N^3}-\frac{11\,719}{45N^4}+\frac{20\,953}{30N^5}-\frac{6\,970\,693}{3780N^6}+\frac{244\,171}{45N^7}
-\frac{1\,195\,407\,011}{75\,600N^8}\biggr)\zeta_2+\biggl(\frac{464}{9}-\frac{44}{N}+\frac{160}{3N^2}-\frac{50}{N^3}+\frac{149}{3N^4}-\frac{50}{N^5}
+\frac{3160}{63N^6}-\frac{50}{N^7}+\frac{299}{6N^8}\biggr)\zeta_3-\frac{5465}{72}+\frac{17\,579}{108N}-\frac{30\,811}{324N^2}+\frac{181}{72N^3}
+\frac{19\,531}{96N^4}-\frac{11\,910\,503}{13\,500N^5}+\frac{233\,947\,499}{85\,050N^6}
-\frac{48\,217\,963\,501}{5\,556\,600N^7}+\frac{979\,578\,746\,771}{38\,102\,400N^8}\biggr)+O\left(\ln^3(\bar{N})\frac{1}{N^9}\right). \tag{B.2}
$$

Appendix C. The heavy flavor Wilson coefficients in x-space

The x-space representations of the heavy flavor Wilson coefficients can be expressed in terms of harmonic polylogarithms to 2-loop order. Also here arguments x will not be written explicitly. They read:

$$
\begin{aligned}
L_{2,q}^{W^++W^-,\mathrm{NS},(2)} ={}& C_F T_F\biggl\{\frac{265}{9}\delta(1-x)+\biggl[\frac{16H_{0,1}}{3(1-x)}+\frac{8{H_0}^2}{1-x}+\frac{16H_1H_0}{3(1-x)}+\frac{268H_0}{9(1-x)}\\
&+\frac{8{H_1}^2}{3(1-x)}+\frac{116H_1}{9(1-x)}-\frac{32\zeta_2}{3(1-x)}+\frac{718}{27(1-x)}\biggr]_+-\frac{8}{3}(x+1)H_{0,1}\\
&-(4x+4){H_0}^2-\frac{8}{9}(34x+19)H_0-\frac{8}{3}(x+1)H_1H_0-\frac{4}{3}(x+1){H_1}^2\\
&-\frac{8}{9}(17x+8)H_1+\frac{16}{3}(x+1)\zeta_2-\frac{1244x}{27}-\frac{272}{27}\biggr\}\\
&+C_FT_F\ln\left(\frac{m^2}{Q^2}\right)\biggl\{\frac{2}{3}\delta(1-x)+\biggl[\frac{16H_0}{3(1-x)}+\frac{80}{9(1-x)}\biggr]_+\\
&-\frac{8}{3}(x+1)H_0-\frac{88x}{9}+\frac{8}{9}\biggr\}+C_FT_F\ln^2\left(\frac{m^2}{Q^2}\right)\biggl\{2\delta(1-x)\\
&+\biggl[\frac{8}{3(1-x)}\biggr]_+-\frac{4x}{3}-\frac{4}{3}\biggr\},
\end{aligned} \tag{C.1}
$$

$$
\begin{aligned}
H_{2,q}^{W^++W^-,\mathrm{NS},(2)} ={}& C_F^2\biggl\{\biggl(\frac{64\zeta_2^2}{5}+8\zeta_2-72\zeta_3+\frac{331}{8}\biggr)\delta(1-x)+\biggl[-\frac{8{H_0}^3}{3(1-x)}\\
&-\frac{12H_1{H_0}^2}{1-x}-\frac{3{H_0}^2}{1-x}-\frac{32{H_1}^2H_0}{1-x}+\frac{48\zeta_2H_0}{1-x}-\frac{36H_1H_0}{1-x}\\
&+\frac{48H_{0,-1}H_0}{1-x}-\frac{24H_{0,1}H_0}{1-x}+\frac{61H_0}{1-x}-\frac{8{H_1}^3}{1-x}-\frac{18{H_1}^2}{1-x}+\frac{24\zeta_2}{1-x}\\
&+\frac{64\zeta_3}{1-x}+\frac{16\zeta_2H_1}{1-x}+\frac{27H_1}{1-x}+\frac{16H_1H_{0,1}}{1-x}+\frac{12H_{0,1}}{1-x}-\frac{96H_{0,0,-1}}{1-x}\\
&+\frac{24H_{0,0,1}}{1-x}-\frac{24H_{0,1,1}}{1-x}+\frac{51}{2(1-x)}\biggr]_++\left(x+5-\frac{4}{x+1}\right){H_0}^3\\
&+\left(40x-16+\frac{40}{x+1}\right)H_{-1}{H_0}^2+(10-14x)H_1{H_0}^2
\end{aligned}
$$

$$
\begin{aligned}
&+\left(-56x+8-\frac{32}{x+1}\right)H_{-1}{}^2H_0+16(x+1)H_1{}^2H_0\\
&+\left(\frac{72x^3}{5}-2x+12\right)H_0{}^2\\
&+\left(\frac{144x^2}{5}-\frac{502x}{5}-\frac{132}{5}-\frac{16}{x+1}-\frac{16}{5x}\right)H_0\\
&+\left(-24x-40+\frac{16}{x+1}\right)\zeta_2H_0\\
&+\left(-\frac{144x^3}{5}+40x+72+\frac{16}{5x^2}\right)H_{-1}H_0+32(x+1)H_1H_0\\
&+\left(-80x-\frac{32}{x+1}\right)H_{0,-1}H_0+(56x+8)H_{0,1}H_0+4(x+1)H_1{}^3\\
&+\frac{144x^2}{5}+(18x+14)H_1{}^2-\frac{461x}{5}+\left(-\frac{144x^3}{5}-8x-32\right)\zeta_2\\
&+\left(72x-64+\frac{56}{x+1}\right)\zeta_3+\left(-72x-\frac{64}{x+1}+24\right)\zeta_2H_{-1}\\
&+(16-68x)H_1+(32x-16)\zeta_2H_1+\left(\frac{144x^3}{5}-40x\right.\\
&\left.-72-\frac{16}{5x^2}\right)H_{0,-1}+\left(112x-16+\frac{64}{x+1}\right)H_{-1}H_{0,-1}\\
&+16xH_{0,1}+\left(16x-16+\frac{32}{x+1}\right)H_{-1}H_{0,1}\\
&-8(x+1)H_1H_{0,1}+\left(-112x+16-\frac{64}{x+1}\right)H_{0,-1,-1}\\
&+\left(-16x+16-\frac{32}{x+1}\right)H_{0,-1,1}+\left(80x+32-\frac{16}{x+1}\right)H_{0,0,-1}\\
&+\left(-60x+4-\frac{16}{x+1}\right)H_{0,0,1}+\left(-16x+16-\frac{32}{x+1}\right)H_{0,1,-1}\\
&\left.+16(x+1)H_{0,1,1}+\frac{16}{5x}-\frac{124}{5}\right\}+C_AC_F\left\{\left(-\frac{32\zeta_2^2}{5}-4\zeta_2\right.\right.\\
&\left.+54\zeta_3-\frac{5465}{72}\right)\delta(1-x)+\left[-\frac{2H_0{}^3}{1-x}-\frac{8H_1H_0{}^2}{1-x}-\frac{55H_0{}^2}{3(1-x)}\right.\\
&+\frac{4H_1{}^2H_0}{1-x}+\frac{8\zeta_2H_0}{1-x}-\frac{44H_1H_0}{3(1-x)}-\frac{24H_{0,-1}H_0}{1-x}+\frac{16H_{0,1}H_0}{1-x}\\
&-\frac{239H_0}{3(1-x)}-\frac{22H_1{}^2}{3(1-x)}+\frac{88\zeta_2}{3(1-x)}+\frac{4\zeta_3}{1-x}+\frac{24\zeta_2H_1}{1-x}\\
&-\frac{367H_1}{9(1-x)}-\frac{16H_1H_{0,1}}{1-x}-\frac{44H_{0,1}}{3(1-x)}+\frac{48H_{0,0,-1}}{1-x}-\frac{24H_{0,0,1}}{1-x}\\
&\left.+\frac{24H_{0,1,1}}{1-x}-\frac{3155}{54(1-x)}\right]_++\left(2x+\frac{2}{x+1}\right)H_0{}^3
\end{aligned}
$$

$$
\begin{aligned}
&+\left(-\frac{36x^3}{5}+\frac{115x}{6}+\frac{55}{6}\right)H_0{}^2+\left(-20x+8-\frac{20}{x+1}\right)H_{-1}H_0{}^2\\
&+(14x+2)H_1H_0{}^2+\left(28x-4+\frac{16}{x+1}\right)H_{-1}{}^2H_0\\
&-2(x+1)H_1{}^2H_0+\left(-\frac{72x^2}{5}+\frac{1693x}{15}+\frac{8}{x+1}+\frac{583}{15}+\frac{8}{5x}\right)H_0\\
&+\left(-8x-\frac{8}{x+1}\right)\zeta_2H_0+\left(\frac{72x^3}{5}-20x-36-\frac{8}{5x^2}\right)H_{-1}H_0\\
&+\frac{22}{3}(x+1)H_1H_0+\left(40x+\frac{16}{x+1}\right)H_{0,-1}H_0-(28x+4)H_{0,1}H_0\\
&-\frac{72x^2}{5}+\frac{11}{3}(x+1)H_1{}^2+\frac{17\,626x}{135}+\left(\frac{72x^3}{5}-\frac{104x}{3}-\frac{44}{3}\right)\zeta_2\\
&+\left(-56x+12-\frac{28}{x+1}\right)\zeta_3+\left(36x-12+\frac{32}{x+1}\right)\zeta_2H_{-1}\\
&+\frac{4}{9}(167x+14)H_1-(32x+8)\zeta_2H_1+\left(-\frac{72x^3}{5}+20x+36\right.\\
&\left.+\frac{8}{5x^2}\right)H_{0,-1}+\left(-56x+8-\frac{32}{x+1}\right)H_{-1}H_{0,-1}\\
&+\frac{22}{3}(x+1)H_{0,1}+\left(-8x+8-\frac{16}{x+1}\right)H_{-1}H_{0,1}+8(x+1)H_1H_{0,1}\\
&+\left(56x-8+\frac{32}{x+1}\right)H_{0,-1,-1}+\left(8x-8+\frac{16}{x+1}\right)H_{0,-1,1}\\
&+\left(-40x-16+\frac{8}{x+1}\right)H_{0,0,-1}+\left(36x+4+\frac{8}{x+1}\right)H_{0,0,1}\\
&\left.+\left(8x-8+\frac{16}{x+1}\right)H_{0,1,-1}-12(x+1)H_{0,1,1}-\frac{8}{5x}+\frac{3709}{135}\right\}\\
&+\ln^2\left(\frac{m^2}{Q^2}\right)C_FT_F\left(+\left[\frac{8}{3(1-x)}\right]_++2\delta(1-x)-\frac{4x}{3}-\frac{4}{3}\right)\\
&+\ln\left(\frac{m^2}{Q^2}\right)C_FT_F\left(-\frac{88x}{9}+\frac{2}{3}\delta(1-x)-\frac{8}{3}(x+1)H_0+\frac{8}{9}\right.\\
&\left.+\left[\frac{16H_0}{3(1-x)}+\frac{80}{9(1-x)}\right]_+\right)+C_FT_F\Big((-4x-4)H_0{}^2\\
&-\frac{8}{9}(34x+19)H_0-\frac{8}{3}(x+1)H_1H_0-\frac{4}{3}(x+1)H_1{}^2\\
&-\frac{1244x}{27}+\frac{16}{3}(x+1)\zeta_2+\frac{265}{9}\delta(1-x)-\frac{8}{9}(17x+8)H_1\\
&-\frac{8}{3}(x+1)H_{0,1}+\left[+\frac{8H_0{}^2}{1-x}+\frac{16H_1H_0}{3(1-x)}+\frac{268H_0}{9(1-x)}+\frac{8H_1{}^2}{3(1-x)}\right.\\
&\left.-\frac{32\zeta_2}{3(1-x)}+\frac{116H_1}{9(1-x)}+\frac{16H_{0,1}}{3(1-x)}+\frac{718}{27(1-x)}\right]_+-\frac{272}{27}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+ n_f C_F T_F\biggl(-\frac{10}{3}(x+1)H_0{}^2 - \frac{4}{3}(19x+13)H_0 - \frac{8}{3}(x+1)H_1 H_0 \\
&- \frac{4}{3}(x+1)H_1{}^2 - \frac{976x}{27} + \frac{16}{3}(x+1)\zeta_2 + \frac{457}{18}\delta(1-x) \\
&- \frac{8}{9}(17x+8)H_1 - \frac{8}{3}(x+1)H_{0,1} - \frac{316}{27} + \biggl[\frac{20H_0{}^2}{3(1-x)} + \frac{16H_1H_0}{3(1-x)} \\
&+ \frac{76H_0}{3(1-x)} + \frac{8H_1{}^2}{3(1-x)} - \frac{32\zeta_2}{3(1-x)} + \frac{116H_1}{9(1-x)} \\
&+ \frac{16H_{0,1}}{3(1-x)} + \frac{494}{27(1-x)}\biggr]_+\biggr),
\end{aligned}
\tag{C.2}
$$

$$
\begin{aligned}
H_{2,q}^{W^+-W^-,\mathrm{NS},(2)} = {} & H_{2,q}^{W^++W^-,\mathrm{NS},(2)} + C_F(C_F - C_A/2)\biggl\{\biggl(-\frac{144x^3}{5} + 96x^2 + \frac{16}{5x^2} \\
&+ 64x + 64\biggr)H_{0,-1} + \biggl(32x - 32 + \frac{64}{x+1}\biggr)H_0 H_{0,-1} \\
&+ \biggl(-224x + 32 - \frac{128}{x+1}\biggr)H_{-1}H_{0,-1} \\
&+ \biggl(-32x + 32 - \frac{64}{x+1}\biggr)H_{-1}H_{0,1} \\
&+ 16(x+1)H_{0,1} + \biggl(224x - 32 + \frac{128}{x+1}\biggr)H_{0,-1,-1} \\
&+ \biggl(32x - 32 + \frac{64}{x+1}\biggr)H_{0,-1,1} + \biggl(96x + \frac{32}{x+1}\biggr)H_{0,0,-1} \\
&+ \biggl(16x - 16 + \frac{32}{x+1}\biggr)H_{0,0,1} + \biggl(32x - 32 + \frac{64}{x+1}\biggr)H_{0,1,-1} \\
&+ \biggl(-\frac{144x^2}{5} + \frac{292x}{5} + \frac{32}{x+1} - \frac{28}{5} + \frac{16}{5x}\biggr)H_0 + \biggl(-\frac{72x^3}{5} + 48x^2 \\
&+ 32x + 8\biggr)H_0{}^2 + \biggl(\frac{144x^3}{5} - 96x^2 - \frac{16}{5x^2} - 64x - 64\biggr)H_{-1}H_0 \\
&+ \biggl(-16x + 16 - \frac{32}{x+1}\biggr)\zeta_2 H_0 + \biggl(144x - 48 + \frac{128}{x+1}\biggr)\zeta_2 H_{-1} \\
&+ \biggl(4x - 4 + \frac{8}{x+1}\biggr)H_0{}^3 + \biggl(-80x + 32 - \frac{80}{x+1}\biggr)H_{-1}H_0{}^2 \\
&+ \biggl(112x - 16 + \frac{64}{x+1}\biggr)H_{-1}{}^2 H_0 - 32(x-1)H_1 + \biggl(\frac{144x^3}{5} - 96x^2 \\
&- 56x - 8\biggr)\zeta_2 + \biggl(-136x + 40 - \frac{112}{x+1}\biggr)\zeta_3 - \frac{144x^2}{5} \\
&- \frac{164x}{5} - \frac{16}{5x} + \frac{324}{5}\biggr\},
\end{aligned}
\tag{C.3}
$$

$$
H_{2,q}^{W,\mathrm{PS},(2)} = C_F T_F \ln^2\biggl(\frac{m^2}{Q^2}\biggr)\biggl\{(-4x-4)H_0 + \frac{8x^2}{3} + 2x - \frac{8}{3x} - 2\biggr\}
$$

$$
\begin{aligned}
&+C_F T_F \ln\left(\frac{m^2}{Q^2}\right)\left\{\left(-\frac{32x^2}{3}-20x-4\right)H_0+(4x+4)H_0{}^2+\frac{224x^2}{9}\right.\\
&\left.-24x+8-\frac{80}{9x}\right\}+C_F T_F\left\{\left(\frac{32x^2}{3}+32x+\frac{32}{3x}+32\right)H_{0,-1}\right.\\
&+\left(-16x^2+12x-\frac{16}{x}-12\right)H_{0,1}+(24x+24)H_0 H_{0,1}\\
&+(-16x-16)H_{0,0,1}+(16x+16)H_{0,1,1}\\
&+\left(-\frac{56x^2}{3}+35x-1\right)H_0{}^2+\left(-\frac{160x^2}{3}-\frac{220x}{3}+\frac{308}{3}\right)H_0\\
&+\left(-\frac{32x^2}{3}-32x-\frac{32}{3x}-32\right)H_{-1}H_0+\left(-16x^2-12x+\frac{16}{x}+12\right)H_1 H_0\\
&+\left(-\frac{16x^2}{3}-4x+\frac{16}{3x}+4\right)H_1{}^2+\left(\frac{64x^2}{9}-\frac{160x}{3}-\frac{208}{9x}+\frac{208}{3}\right)H_1\\
&+(-32x-32)\zeta_2 H_0+(6x+6)H_0{}^3+\left(32x^2-32x-\frac{32}{3x}\right)\zeta_2+\frac{1696x^2}{27}\\
&\left.-\frac{1030x}{9}+\frac{464}{27x}+\frac{310}{9}\right\},
\end{aligned}
\tag{C.4}
$$

$$
\begin{aligned}
L_{2,g}^{W,(2)}=\ln\left(\frac{m^2}{Q^2}\right)T_F^2&\left\{\left(-\frac{32x^2}{3}+\frac{32x}{3}-\frac{16}{3}\right)H_0+\left(-\frac{32x^2}{3}+\frac{32x}{3}-\frac{16}{3}\right)H_1\right.\\
&\left.-\frac{128x^2}{3}+\frac{128x}{3}-\frac{16}{3}\right\},
\end{aligned}
\tag{C.5}
$$

$$
\begin{aligned}
H_{2,g}^{W,(2)}=\ln^2\left(\frac{m^2}{Q^2}\right)&\left\{T_F^2\left(-\frac{16x^2}{3}+\frac{16x}{3}-\frac{8}{3}\right)+C_F T_F\big(4x-1+\big(-8x^2+4x-2\big)H_0\right.\\
&+\big(-8x^2+8x-4\big)H_1\big)+C_A T_F\left(\frac{62x^2}{3}-16x-2-\frac{8}{3x}+(-16x-4)H_0\right.\\
&\left.\left.+\big(8x^2-8x+4\big)H_1\right)\right\}+\ln\left(\frac{m^2}{Q^2}\right)\left\{T_F^2\left(-\frac{128x^2}{3}+\frac{128x}{3}-\frac{16}{3}\right.\right.\\
&\left.+\left(-\frac{32x^2}{3}+\frac{32x}{3}-\frac{16}{3}\right)H_0+\left(-\frac{32x^2}{3}+\frac{32x}{3}-\frac{16}{3}\right)H_1\right)\\
&+C_A T_F\left(\frac{872x^2}{9}-100x+8-\frac{80}{9x}+16x\zeta_2+(8x+4)H_0{}^2\right.\\
&+\big(8x^2-8x+4\big)H_1{}^2+\left(-\frac{176x^2}{3}-32x-4\right)H_0\\
&\left.+\big(16x^2+16x+8\big)H_{-1}H_0+\big(16x^2-16x\big)H_1+\big(-16x^2-16x-8\big)H_{0,-1}\right)\\
&+C_F T_F\big(-8x^2+34x-18+\big(-16x^2+8x-4\big)H_0{}^2\\
&+\big(-16x^2+16x-8\big)H_1{}^2+\big(32x^2-24x+12\big)\zeta_2+\big(-40x^2+24x-4\big)H_0\\
&\left.+\big(-40x^2+48x-14\big)H_1+\big(-32x^2+32x-16\big)H_0 H_1+(4-8x)H_{0,1}\big)\right\}
\end{aligned}
$$

$$
\begin{aligned}
&+C_A T_F\Biggl\{\left(\frac{52x}{3}+6\right)H_0{}^3+\left(-\frac{365x^2}{3}+180x-1\right)H_0{}^2\\
&+\left(28x^2+20x+10\right)H_{-1}H_0{}^2+\left(-16x^2+24x-12\right)H_1H_0{}^2\\
&+\left(-8x^2+8x+4\right)H_{-1}{}^2H_0+\left(-28x^2+28x-14\right)H_1{}^2H_0\\
&+\left(-\frac{1660x^2}{3}+\frac{1082x}{3}+\frac{320}{3}\right)H_0+\left(32x^2-128x-16\right)\zeta_2H_0\\
&+\left(\frac{184x^2}{3}+8x-48-\frac{32}{3x}\right)H_{-1}H_0+\left(-222x^2+192x-2+\frac{16}{x}\right)H_1H_0\\
&+\left(-24x^2-40x-20\right)H_{0,-1}H_0+(80x+32)H_{0,1}H_0\\
&+\left(-\frac{4x^2}{3}+\frac{4x}{3}-\frac{2}{3}\right)H_1{}^3-\frac{5810x^2}{27}+\frac{1202x}{9}+\frac{472}{9}+\frac{464}{27x}\\
&+\left(-\frac{229x^2}{3}+68x-5+\frac{16}{3x}\right)H_1{}^2+\left(294x^2-280x+16-\frac{32}{3x}\right)\zeta_2\\
&+\left(48x^2-40x+24\right)\zeta_3+\left(-40x^2-24x-12\right)\zeta_2H_{-1}\\
&+\left(-\frac{3068x^2}{9}+\frac{884x}{3}+\frac{118}{3}-\frac{208}{9x}\right)H_1+\left(16x^2-32x+16\right)\zeta_2H_1\\
&+\left(-\frac{184x^2}{3}-8x+48+\frac{32}{3x}\right)H_{0,-1}+\left(16x^2-16x-8\right)H_{-1}H_{0,-1}\\
&+\left(-72x^2+96x-14-\frac{16}{x}\right)H_{0,1}+\left(32x^2+32x+16\right)H_{-1}H_{0,1}\\
&+\left(32x^2-32x+16\right)H_1H_{0,1}+\left(-16x^2+16x+8\right)H_{0,-1,-1}\\
&+\left(-32x^2-32x-16\right)H_{0,-1,1}+\left(-8x^2+40x+20\right)H_{0,0,-1}\\
&+(-80x-24)H_{0,0,1}+\left(-32x^2-32x-16\right)H_{0,1,-1}\\
&+\left(-40x^2+104x-4\right)H_{0,1,1}\Biggr\}+C_FT_F\Biggl\{\left(-12x^2+6x-3\right)H_0{}^3\\
&+\left(-\frac{96x^3}{5}-62x^2+\frac{26x}{3}-\frac{7}{2}\right)H_0{}^2+\left(16x^2+32x+16\right)H_{-1}H_0{}^2\\
&+\left(-20x^2+4x-2\right)H_1H_0{}^2+\left(-32x^2-64x-32\right)H_{-1}{}^2H_0\\
&+\left(-32x^2+32x-16\right)H_1{}^2H_0+\left(-\frac{552x^2}{5}+\frac{181x}{5}-\frac{592}{15}-\frac{16}{15x}\right)H_0\\
&+\left(96x^2-64x+32\right)\zeta_2H_0+\left(\frac{192x^3}{5}+\frac{128x}{3}+96+\frac{16}{15x^2}\right)H_{-1}H_0\\
&+\left(-124x^2+136x-50\right)H_1H_0+\left(32x^2-64x+32\right)H_{0,-1}H_0\\
&+\left(-24x^2+48x-24\right)H_{0,1}H_0+\left(-\frac{44x^2}{3}+\frac{44x}{3}-\frac{22}{3}\right)H_1{}^3\\
&+\frac{128x^2}{5}+\frac{273x}{5}+\frac{16}{15x}-\frac{1099}{15}+\left(-78x^2+84x-24\right)H_1{}^2
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{192x^3}{5}+156x^2-\frac{280x}{3}+28\right)\zeta_2+\left(152x^2+8x+60\right)\zeta_3\\
&+\left(-32x^2-64x-32\right)\zeta_2 H_{-1}+\left(-72x^2+106x-28\right)H_1\\
&+\left(64x^2-32x+16\right)\zeta_2 H_1+\left(-\frac{192x^3}{5}-\frac{128x}{3}-96-\frac{16}{15x^2}\right)H_{0,-1}\\
&+\left(64x^2+128x+64\right)H_{-1}H_{0,-1}+\left(22-32x^2\right)H_{0,1}\\
&+\left(-32x^2+32x-16\right)H_1 H_{0,1}+\left(-64x^2-128x-64\right)H_{0,-1,-1}\\
&+\left(-96x^2+64x-96\right)H_{0,0,-1}+\left(-8x^2-40x+20\right)H_{0,0,1}\\
&+\left(40x^2-56x+28\right)H_{0,1,1}\Bigg\}, \qquad \text{(C.6)}
\end{aligned}
$$

$$
\begin{aligned}
L_{3,q}^{W^++W^-,\mathrm{NS},(2)} ={}& C_F T_F \ln^2\left(\frac{m^2}{Q^2}\right)\left\{2\delta(1-x)+\left[\frac{8}{3(1-x)}\right]_+-\frac{4x}{3}-\frac{4}{3}\right\}\\
&+C_F T_F \ln\left(\frac{m^2}{Q^2}\right)\left\{\frac{2}{3}\delta(1-x)+\left[\frac{16H_0}{3(1-x)}+\frac{80}{9(1-x)}\right]_+\right.\\
&\left.+\left(-\frac{8x}{3}-\frac{8}{3}\right)H_0-\frac{88x}{9}+\frac{8}{9}\right\}+C_F T_F\left\{\left[+\frac{16H_{0,1}}{3(1-x)}+\frac{8{H_0}^2}{1-x}\right.\right.\\
&+\frac{16H_1H_0}{3(1-x)}+\frac{268H_0}{9(1-x)}+\frac{8{H_1}^2}{3(1-x)}+\frac{116H_1}{9(1-x)}-\frac{32\zeta_2}{3(1-x)}\\
&\left.+\frac{718}{27(1-x)}\right]_+-\frac{8}{3}(x+1)H_{0,1}+\frac{265}{9}\delta(1-x)+(-4x-4){H_0}^2\\
&+\left(-\frac{224x}{9}-\frac{104}{9}\right)H_0+\left(-\frac{8x}{3}-\frac{8}{3}\right)H_1H_0-\frac{872x}{27}-\frac{188}{27}\\
&\left.+\left(-\frac{4x}{3}-\frac{4}{3}\right){H_1}^2+\left(-\frac{112x}{9}-\frac{40}{9}\right)H_1+\left(\frac{16x}{3}+\frac{16}{3}\right)\zeta_2\right\}, \qquad \text{(C.7)}
\end{aligned}
$$

$$
\begin{aligned}
H_{3,q}^{W^++W^-,\mathrm{NS},(2)} ={}& H_{2,q}^{W^++W^-,\mathrm{NS},(2)}+C_F^2\left\{\left(4x-4+\frac{8}{x+1}\right){H_0}^3+\left(-\frac{72x^3}{5}+8x^2\right.\right.\\
&\left.+28x+12\right){H_0}^2+\left(-56x+40-\frac{80}{x+1}\right)H_{-1}{H_0}^2\\
&+(24x-8)H_1{H_0}^2+\left(64x-32+\frac{64}{x+1}\right){H_{-1}}^2H_0\\
&+\left(-\frac{144x^2}{5}+\frac{192x}{5}-\frac{68}{5}+\frac{32}{x+1}+\frac{16}{5x}\right)H_0\\
&+\left(-16x+16-\frac{32}{x+1}\right)\zeta_2H_0+\left(\frac{144x^3}{5}-16x^2-48x\right.\\
&\left.-80-\frac{16}{x}-\frac{16}{5x^2}\right)H_{-1}H_0-8(x+1)H_1H_0\\
&+\left(80x-48+\frac{64}{x+1}\right)H_{0,-1}H_0+(16-48x)H_{0,1}H_0
\end{aligned}
$$

$$
\begin{aligned}
& - 4(x+1) H_1{}^2 + \left(\frac{144x^3}{5} - 16x^2 - 44x - 12\right)\zeta_2 \\
& + \left(-136x + 72 - \frac{112}{x+1}\right)\zeta_3 + \left(96x - 64 + \frac{128}{x+1}\right)\zeta_2 H_{-1} \\
& + (46x - 10) H_1 + (16 - 48x)\zeta_2 H_1 + \left(-\frac{144x^3}{5} + 16x^2 + 48x \right.\\
& \left. + 80 + \frac{16}{x} + \frac{16}{5x^2}\right) H_{0,-1} + \left(-128x + 64 - \frac{128}{x+1}\right) H_{-1} H_{0,-1} \\
& + 12(x+1) H_{0,1} + \left(-32x + 32 - \frac{64}{x+1}\right) H_{-1} H_{0,1} \\
& + \left(128x - 64 + \frac{128}{x+1}\right) H_{0,-1,-1} + \left(32x - 32 + \frac{64}{x+1}\right) H_{0,-1,1} \\
& + \left(-48x + 16 + \frac{32}{x+1}\right) H_{0,0,-1} + \left(64x - 32 + \frac{32}{x+1}\right) H_{0,0,1} \\
& + \left(32x - 32 + \frac{64}{x+1}\right) H_{0,1,-1} - \frac{144x^2}{5} + \frac{561x}{5} - \frac{231}{5} - \frac{16}{5x}\Bigg\} \\
& + C_A C_F \Bigg\{ \left(-2x + 2 - \frac{4}{x+1}\right) H_0{}^3 + \left(\frac{36x^3}{5} - 4x^2 - 16x - 8\right) H_0{}^2 \\
& + \left(28x - 20 + \frac{40}{x+1}\right) H_{-1} H_0{}^2 + (4 - 12x) H_1 H_0{}^2 \\
& + \left(-32x + 16 - \frac{32}{x+1}\right) H_{-1}{}^2 H_0 + \left(\frac{72x^2}{5} - \frac{478x}{15} + \frac{2}{15} \right.\\
& \left. - \frac{16}{x+1} - \frac{8}{5x}\right) H_0 + \left(8x - 8 + \frac{16}{x+1}\right)\zeta_2 H_0 \\
& + \left(-\frac{72x^3}{5} + 8x^2 + 24x + 40 + \frac{8}{x} + \frac{8}{5x^2}\right) H_{-1} H_0 \\
& + \left(-40x + 24 - \frac{32}{x+1}\right) H_{0,-1} H_0 + (24x - 8) H_{0,1} H_0 \\
& + \left(-\frac{72x^3}{5} + 8x^2 + 28x + 12\right)\zeta_2 + \left(68x - 36 + \frac{56}{x+1}\right)\zeta_3 \\
& + \left(-48x + 32 - \frac{64}{x+1}\right)\zeta_2 H_{-1} - \frac{2}{3}(47x - 1) H_1 + (24x - 8)\zeta_2 H_1 \\
& + \left(\frac{72x^3}{5} - 8x^2 - 24x - 40 - \frac{8}{x} - \frac{8}{5x^2}\right) H_{0,-1} \\
& + \left(64x - 32 + \frac{64}{x+1}\right) H_{-1} H_{0,-1} - 8(x+1) H_{0,1} \\
& + \left(16x - 16 + \frac{32}{x+1}\right) H_{-1} H_{0,1} + \left(-64x + 32 - \frac{64}{x+1}\right) H_{0,-1,-1} \\
& + \left(-16x + 16 - \frac{32}{x+1}\right) H_{0,-1,1} + \left(24x - 8 - \frac{16}{x+1}\right) H_{0,0,-1}
\end{aligned}
$$

$$
\begin{aligned}
&+\left(-32x+16-\frac{16}{x+1}\right)H_{0,0,1}+\left(-16x+16-\frac{32}{x+1}\right)H_{0,1,-1}\\
&+\frac{72x^2}{5}-\frac{3517x}{45}+\frac{8}{5x}+\frac{647}{45}\biggr\}+C_FT_F\biggl\{\frac{124x}{9}+\frac{28}{9}\\
&+\frac{16}{3}(x+1)H_0+\frac{8}{3}(x+1)H_1\biggr\}\\
&+n_fC_FT_F\biggl\{\frac{124x}{9}+\frac{28}{9}+\frac{16}{3}(x+1)H_0+\frac{8}{3}(x+1)H_1\biggr\}, \qquad \text{(C.8)}
\end{aligned}
$$

$$
\begin{aligned}
H_{3,q}^{W^+-W^-,\mathrm{NS},(2)}={}&H_{3,q}^{W^++W^-,\mathrm{NS},(2)}+C_F(C_F-C_A/2)\biggl\{\left(-16x^2-\frac{16}{x}\right)H_{0,-1}\\
&+\left(-32x+32-\frac{64}{x+1}\right)H_0H_{0,-1}+\left(32x-96+\frac{128}{x+1}\right)H_{-1}H_{0,-1}\\
&-16(x+1)H_{0,1}+\left(32x-32+\frac{64}{x+1}\right)H_{-1}H_{0,1}\\
&+\left(-32x+96-\frac{128}{x+1}\right)H_{0,-1,-1}+\left(-32x+32-\frac{64}{x+1}\right)H_{0,-1,1}\\
&+\left(32-\frac{32}{x+1}\right)H_{0,0,-1}+\left(-16x+16-\frac{32}{x+1}\right)H_{0,0,1}\\
&+\left(-32x+32-\frac{64}{x+1}\right)H_{0,1,-1}+\left(-8x^2-8x-16\right)H_0{}^2\\
&+\left(16x^2+\frac{16}{x}\right)H_{-1}H_0+\left(16x-16+\frac{32}{x+1}\right)\zeta_2H_0\\
&+\left(-48x+80-\frac{128}{x+1}\right)\zeta_2H_{-1}+\left(-4x+4-\frac{8}{x+1}\right)H_0{}^3\\
&+\left(32x-48+\frac{80}{x+1}\right)H_{-1}H_0{}^2+\left(-16x+48-\frac{64}{x+1}\right)H_{-1}{}^2H_0\\
&+\left(60x+28-\frac{32}{x+1}\right)H_0+32(x-1)H_1+\left(16x^2+8x+24\right)\zeta_2\\
&+\left(40x-72+\frac{112}{x+1}\right)\zeta_3-60x+60\biggr\}, \qquad \text{(C.9)}
\end{aligned}
$$

$$
\begin{aligned}
H_{3,q}^{W,\mathrm{PS},(2)}={}&C_FT_F\ln^2\left(\frac{m^2}{Q^2}\right)\biggl\{(4x+4)H_0-\frac{8x^2}{3}-2x+\frac{8}{3x}+2\biggr\}\\
&+C_FT_F\ln\left(\frac{m^2}{Q^2}\right)\biggl\{\left(\frac{32x^2}{3}+20x+4\right)H_0+(-4x-4)H_0{}^2-\frac{224x^2}{9}+24x\\
&-8+\frac{80}{9x}\biggr\}+C_FT_F\biggl\{\left(-\frac{16x^2}{3}-4x+\frac{16}{3x}+4\right)H_{0,1}+(-8x-8)H_0H_{0,1}\\
&+(16x+16)H_{0,0,1}+\left(-\frac{8x^2}{3}-5x-1\right)H_0{}^2+\left(\frac{224x^2}{9}+\frac{44x}{3}+\frac{28}{3}\right)H_0\\
&+\left(\frac{16x^2}{3}+4x-\frac{16}{3x}-4\right)H_1H_0+\left(\frac{2x}{3}+\frac{2}{3}\right)H_0{}^3-\frac{800x^2}{27}
\end{aligned}
$$

$$+ (-16x-16)\zeta_3 + \frac{62x}{3} + \frac{224}{27x} + \frac{2}{3}\Bigg\}, \tag{C.10}$$

$$\begin{aligned}
H_{3,g}^{W,(2)} = {} & \ln^2\left(\frac{m^2}{Q^2}\right)\Bigg\{T_F^2\left(\frac{16x^2}{3} - \frac{16x}{3} + \frac{8}{3}\right) + C_A T_F\Bigg(-\frac{62x^2}{3} + 16x + (16x+4)H_0 \\
& + \big(-8x^2+8x-4\big)H_1 + 2 + \frac{8}{3x}\Bigg) + C_F T_F\big(-4x + \big(8x^2-4x+2\big)H_0 \\
& + \big(8x^2-8x+4\big)H_1 + 1\big)\Bigg\} + \ln\left(\frac{m^2}{Q^2}\right)\Bigg\{C_A T_F\Bigg(-16\zeta_2 x + (-8x-4){H_0}^2 \\
& + \big(-8x^2+8x-4\big){H_1}^2 + \left(\frac{176x^2}{3} + 32x + 4\right)H_0 \\
& + \big(-16x^2-16x-8\big)H_{-1}H_0 + \big(16x-16x^2\big)H_1 + \big(16x^2+16x+8\big)H_{0,-1} \\
& - \frac{872x^2}{9} + 100x - 8 + \frac{80}{9x}\Bigg) + C_F T_F\big(14x^2 - 40x + \big(16x^2-8x+4\big){H_0}^2 \\
& + \big(16x^2-16x+8\big){H_1}^2 + \big(-32x^2+24x-12\big)\zeta_2 + \big(40x^2-28x+6\big)H_0 \\
& + \big(40x^2-48x+14\big)H_1 + \big(32x^2-32x+16\big)H_0H_1 + (8x-4)H_{0,1} + 18\big)\Bigg\} \\
& + C_A T_F\Bigg\{\left(\frac{4x}{3} + \frac{2}{3}\right){H_0}^3 + \left(-\frac{23x^2}{3} - 4x - 1\right){H_0}^2 \\
& + \big(-4x^2-4x-2\big)H_{-1}{H_0}^2 + \big(8x^2+8x+4\big){H_{-1}}^2H_0 \\
& + \big(4x^2-4x+2\big){H_1}^2H_0 + \left(\frac{800x^2}{9} + \frac{86x}{3} + \frac{28}{3}\right)H_0 + \big(-8x^2-8x\big)H_{-1}H_0 \\
& + \left(\frac{130x^2}{3} - 32x - 6 - \frac{16}{3x}\right)H_1H_0 + \big(8x^2+8x+4\big)H_{0,-1}H_0 \\
& + (-32x-8)H_{0,1}H_0 + \left(-\frac{4x^2}{3} + \frac{4x}{3} - \frac{2}{3}\right){H_1}^3 - \frac{3176x^2}{27} \\
& + \big(-5x^2+4x+1\big){H_1}^2 + \frac{314x}{3} + \big(2x^2-8x\big)\zeta_2 + (-56x-16)\zeta_3 \\
& + \big(8x^2+8x+4\big)\zeta_2H_{-1} + \big(-8x^2+8x+2\big)H_1 + \big(8x^2+8x\big)H_{0,-1} \\
& + \big(-16x^2-16x-8\big)H_{-1}H_{0,-1} + \left(-\frac{136x^2}{3} + 32x + 6 + \frac{16}{3x}\right)H_{0,1} \\
& + \big(16x^2+16x+8\big)H_{0,-1,-1} + \big(-8x^2-8x-4\big)H_{0,0,-1} + (64x+16)H_{0,0,1} \\
& + \big(-8x^2+8x-4\big)H_{0,1,1} + \frac{224}{27x} + \frac{2}{3}\Bigg\} + C_F T_F\Bigg\{\left(-\frac{4x^2}{3} + \frac{2x}{3} - \frac{1}{3}\right){H_0}^3 \\
& + \left(-10x^2 + 6x + \frac{1}{2}\right){H_0}^2 + \big(-4x^2+4x-2\big)H_1{H_0}^2 + \big(24x^2+9x+8\big)H_0 \\
& + \big(-20x^2+24x-2\big)H_1H_0 + \big(8x^2-16x+8\big)H_{0,1}H_0 \\
& + \left(\frac{4x^2}{3} - \frac{4x}{3} + \frac{2}{3}\right){H_1}^3 - 40x^2 + \big(6x^2-4x-2\big){H_1}^2 + 41x
\end{aligned}$$

$$
\begin{aligned}
&+\left(-12x^2+24x+4\right)\zeta_2+\left(-8x^2-8x+4\right)\zeta_3+\left(24x^2-26x\right)H_1\\
&+\left(32x^2-48x-2\right)H_{0,1}+\left(-8x^2+24x-12\right)H_{0,0,1}\\
&+\left(8x^2-8x+4\right)H_{0,1,1}-13\Big\}.
\end{aligned}
\tag{C.11}
$$

The $+$-distribution in the above relations is defined by

$$
\int_0^1 dx\left[f(x)\right]_+ g(x)=\int_0^1 dx\, f(x)\left[g(x)-g(1)\right].
\tag{C.12}
$$

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